# Metric Corrections of the Affine Camera

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**Abstract**

Given a general affine camera, we study the problem of finding the closest metric affine camera, where the latter is one of the orthographic, weak-perspective and paraperspective projection models. This problem typically arises in stratified Structure-from-Motion methods such as factorization-based methods. For each type of metric affine camera, we give a closed-form solution and its implementation through an algebraic procedure. Using our algebraic procedure, we can then provide a complete analysis of the problem's generic ambiguity space. This also gives the means to generate the other solutions if any.

**Code release.** The Matlab implementation of our three algebraic procedures has been made available under the GPL licence.

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# Contents

# 1 Introduction

We study the problems of finding the closest orthographic, weak-perspective or paraperspective projection to a general affine camera in the sense of the Frobenius norm. These form three instances of the *metric affine correction* problem class, which we called *orthographic affine correction*, *weak-perspective affine correction* and *paraperspective affine correction*, respectively. The main use of metric affine correction is in Structure-from-Motion by factorization (Poelman and Kanade, 1997; Tomasi and Kanade, 1992) and alternation (Marques and Costeira, 2009). In the factorization algorithm, metric affine correction is the final stage of a three-stage process. In the first stage, a centred measurement matrix is factored into a joint camera matrix and a structure matrix. This factorization represents an affine 3D reconstruction and is defined up to a $(3 \times 3)$ matrix representing an affine change of coordinates. In the second stage, the metric structure of the affine 3D reconstruction is recovered by computing an affine-to-metric upgrade using the metric constraints from the camera model (for instance, the two rows of the orthographic camera must be orthonormal). The metric constraints are redundant, and can thus only be satisfied approximately. This means that, with noise, the upgraded camera factor is not exactly a metric camera factor. In the third stage, metric affine correction must therefore be performed for each camera in order to recover the metric cameras from the upgraded affine cameras. (Tomasi and Kanade, 1992) does factorization with the orthographic camera, while (Poelman and Kanade, 1997) does factorization with the paraperspective camera, but uses a suboptimal metric affine correction procedure, which could thus be replaced by the proposed one. In the alternation algorithm, metric affine correction is the third stage of an iterative three-stage process. The alternation algorithm requires one to provide an initial estimate of the cameras. In the first stage, the structure is computed from the current camera estimates by triangulation. In the second stage, each camera is computed from the current structure estimate by resection. Both stages amount to solve a set of small linear least squares problems. The second stage estimates affine cameras, as it leaves aside the non-linear constraints characterizing each type of metric camera model. In the third stage, metric affine correction is thus performed for each camera in order to recover the metric cameras. These three stages are repeated until convergence is reached. The third stage is fundamental in the alternation algorithm for two reasons. The first reason is that because of noise, similarly to the third stage of the factorization algorithm, the computed general affine cameras are not exactly metric cameras. The second reason is probably more important: without the third stage, an alternation algorithm would converge to an affine, and not to a metric, reconstruction. The third stage indeed introduces the metric constraints into the alternation algorithm. (Marques and Costeira, 2009) does alternation with the weak-perspective camera, and could be readily extended to the paraperspective camera with our correction procedure.

Metric affine correction shares strong similarities with orthonormal Procrustes analysis. Inspired by the derivation of the optimal solution to orthonormal Procrustes analysis (specifically, we follow the derivation in (Bartoli et al., 2013) inspired by (Horn et al., 1988; Schönemann, 1966)), we solve orthographic affine correction and weak-perspective affine correction by a simple algebraic procedure, whose derivation is also fairly simple but does not seem to have appeared in the literature before. We also solve paraperspective affine correction by a simple algebraic procedure. Its derivation is however far more involved. We establish the algebraic procedures and prove their optimality. Our procedures allow us to provide an analysis of the problem's generic ambiguities. These are generic in the sense that they apply to any solution algorithm. Our analysis thus determines cases for which the problem has a unique solution, and cases for which it does not. For the latter, we provide a characterization of the solution space[1] and a means to generate all solutions. Our results on the solution ambiguities are summarized in table 1. Metric affine correction is a set of constrained polynomial optimization problems, to which polynomial optimization methods could be applied. This would however be computationally more expensive by several orders of magnitude than our analytical solutions and would not reveal the problems' intrinsic structure and degenerate cases.

Our input data is an affine projection matrix written as $\mathtt{P} \in \mathbb{R}^{2\times 3}$ (and the direction of projection in the paraperspective case). Our goal is to perform metric affine correction on $\mathtt{P}$. For the orthographic camera, this means finding the camera's rotation, and for the weak-perspective and paraperspective cameras, this means finding the camera's rotation and scale factor. The rank of $\mathtt{P}$ must be two (Hartley and Zisserman,

[1] A space is a set (a simple collection of objects) with some added structure, such as a norm.

2003). A rank of one would mean that all 3D space points would be projected to a single image line; a rank of zero would mean that they would be projected to a single image point. Even if these are not proper projections, the rank of matrix $\mathtt{P}$ may drop to one or zero for near degenerate geometries under the effect of noise. For instance, a rank of one may happen when viewing an object with a strong tilt, while a rank of zero may happen when viewing an object at a distance with a narrow field of view. Analyzing degenerate cases thus tells us what may happen in near degenerate cases. We established that, excluding the case where $\mathtt{P}$ vanishes (which is equivalent to it having a zero rank), the weak-perspective and paraperspective scale is always uniquely recoverable. However, for the three metric affine cameras, the rotation is uniquely recoverable only if $\mathtt{P}$ has full rank, otherwise it has an ambiguity in $\mathbb{SO}_2$.

| *Camera model* | rank($\mathtt{P}$) = 2 | rank($\mathtt{P}$) = 1 | rank($\mathtt{P}$) = 0 |
|---|---|---|---|
| *Orthographic* | | | |
| Rotation | Unique | $\mathbb{SO}_2$ ambiguous | Unrecoverable |
| *Weak-perspective* | | | |
| Scale | Unique | Unique | Unique |
| Rotation | Unique | $\mathbb{SO}_2$ ambiguous | Unrecoverable |
| *Paraperspective* | | | |
| Scale | Unique | Unique | Unique |
| Rotation | Unique | $\mathbb{SO}_2$ ambiguous | Unrecoverable |

Table 1: **Summary of our results on solution uniqueness.** Matrix $\mathtt{P} \in \mathbb{R}^{2\times 3}$ is a known upgraded affine projection matrix whose correction into one of the three listed metric affine camera models is sought. The case rank($\mathtt{P}$) = 2 includes the two sub-cases where the singular values of $\mathtt{P}$ are distinct or equal.

We first give our notation and background in §2. We then solve the metric correction problem for the orthographic, weak-perspective and paraperspective cameras in §§3, 4 and 5 respectively. For each camera model, we first give the correction's cost function and pseudo-code. We then derive the correction procedure based on the Singular Value Decomposition (SVD) and analyze the correction's ambiguities. The details of our analysis of the correction's ambiguities for the paraperspective camera are deferred to appendix A. We finally give experimental results in §6 and conclude in §7.

# 2 Notation and Background

## 2.1 Notation

**General notation.** We use italics for scalar (such as $a$ and $\alpha$), bold fonts for vector (such as $\mathbf{v}$) and typewriter fonts for matrices (such as $\mathtt{A}$). The entries of a vector or matrix are written as in $\mathtt{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$. We use diag to create (block) diagonal matrices. We use double bar fonts for sets (such as $\mathbb{R}$). We use $\mathbb{B}$ to denote a generic binary set with $|\mathbb{B}| = 2$. We have for instance $\{-1, 1\} \equiv \mathbb{B}$. We write vector two-norm as in $\|\mathbf{v}\|_2$ and matrix Frobenius norm as in $\|\mathtt{A}\|_{\mathcal{F}}$. We define $[a, b]_\times \stackrel{\text{def}}{=} \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $\odot$ as the Hadamard (element-wise) product.

**Orthonormal matrices.** We use $\mathbb{O}_d$ for the Lie group of orthonormal matrices[2] and $\mathbb{SO}_d \subset \mathbb{O}_d$ for the Lie group of special orthonormal matrices, with $d \in \{2, 3\}$. For $\mathtt{A} \in \mathbb{O}_d$, $\det(\mathtt{A}) = \pm 1$; for $\mathtt{A} \in \mathbb{SO}_d$, $\det(\mathtt{A}) = 1$. We thus have $\mathbb{O}_d \equiv \mathbb{SO}_d \times \mathbb{B}$. Elements of $\mathbb{SO}_2$ may be parameterized as $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for $\theta \in \mathbb{R}$. Elements

[2] A group is a set associated with an operation called the group law. The set and group law must satisfy closure and associativity, and there must be an identity element and an inverse element for each member of the group. For instance, the group law of $\mathbb{O}_d$ and $\mathbb{SO}_d$ is matrix multiplication, the identity element is the identity matrix in $\mathbb{R}^{d\times d}$ and the inverse is element is obtained by matrix transposition.

of $\mathbb{O}_2$ may be parameterized as $\begin{bmatrix} a\cos\theta & -\sin\theta \\ a\sin\theta & \cos\theta \end{bmatrix}$ for $\theta \in \mathbb{R}$ and $a \in \{-1, 1\}$. This is equivalent to having $\begin{bmatrix} b\cos\mu & -b\sin\mu \\ \sin\mu & \cos\mu \end{bmatrix}$ for some $\mu \in \mathbb{R}$ and $b \in \{-1, 1\}$. For $\mathtt{A} \in \mathbb{O}_2$, $\det(\mathtt{A}) = \det(-\mathtt{A})$, and the variable $a$ is thus required to specify whether $\mathtt{A} \in \mathbb{SO}_2$ (for $a = 1$) or $\mathtt{A} \in \mathbb{O}_2 \backslash \mathbb{SO}_2$ (for $a = -1$). For $\mathtt{A} \in \mathbb{O}_3$ however, $\det(\mathtt{A}) = -\det(-\mathtt{A})$, and $\mathtt{A} \in \mathbb{SO}_3$ can thus be switched to $\mathbb{O}_3 \backslash \mathbb{SO}_3$ by simply negating its entries. We write $\mathbb{P}_2$ for the space of $(2 \times 2)$ permutation matrices defined as $\mathbb{P}_2 \stackrel{\text{def}}{=} \{\mathtt{I}, \tilde{\mathtt{I}}\}$ with $\tilde{\mathtt{I}} \stackrel{\text{def}}{=} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. We have that $\mathbb{P}_2 \subset \mathbb{O}_2$ and $\mathbb{P}_2 \equiv \mathbb{B}$,

**Sub-Stiefel matrices.** A sub-Stiefel set $\mathbb{SS}_{r\times c}$, $1 \le r \le 3$, $1 \le c \le 3$, is formed as the set of $r \times c$ blocks taken from all orthonormal matrices in $\mathbb{O}_3$. Consequently, the Frobenius norm of any element of $\mathbb{SS}_{r\times c}$ is bounded by 1. For instance, $n \in \mathbb{SS}_{1\times 1} \subset \mathbb{R}$ is a scalar such that $|n| \le 1$ and $\mathbf{n} \in \mathbb{SS}_{2\times 1} \subset \mathbb{R}^{2\times 1}$ is a vector such that $\|\mathbf{n}\|_2 \le 1$.

## 2.2 Metric Affine Camera Models

The affine camera is simply defined as a projection which preserves parallelism. The general affine camera is thus represented by a matrix $\mathtt{A} \in \mathbb{R}^{2\times 3}$ for the rotational part and a vector $\mathbf{t} \in \mathbb{R}^{2\times 1}$ for the translational part. More specifically, a point with world coordinates $\mathbf{Q} \in \mathbb{R}^{3\times 1}$ is projected to image coordinates $\mathbf{q} \in \mathbb{R}^{2\times 1}$ as $\mathbf{q} = \mathtt{A}\mathbf{Q} + \mathbf{t}$. We use *metric affine camera* to mean an affine camera which satisfies some additional constraints called the metric constraints. Metric affine cameras are important: they form the basis of many *Shape-from-X* methods, such as Photometric Stereo (Woodham, 1980) and Shape-from-Shading (Horn, 1989), to name a few. The metric affine cameras may be derived from the perspective camera in two ways. First, by increasing the focal length to infinity while back-tracking along the principal ray (Hartley and Zisserman, 2003). Second, by approximating perspective projection to some order (Faugeras et al., 2001).

The orthographic camera is the simplest metric affine camera. An affine camera is orthographic if $\mathtt{A} = \bar{\mathtt{R}}$ with $\bar{\mathtt{R}} \in \mathbb{SS}_{2\times 3}$. In other words, $\mathtt{A}$ must be the leading two rows of a 3D rotation matrix. Geometrically, it rotates the object's points and simply projects them orthographically to the camera's retina. The weak-perspective camera is a zeroth order approximation of the perspective camera. An affine camera is weak-perspective if $\mathtt{A} = \alpha\bar{\mathtt{R}}$ with $\alpha \in \mathbb{R}^+$ and $\bar{\mathtt{R}} \in \mathbb{SS}_{2\times 3}$. In other words, $\mathtt{A}$ must be the leading two rows of a 3D rotation matrix up to a positive rescaling. Geometrically, it follows the orthographic camera, and additionally rescales the projected points about the optical axis. The rescaling takes into account the camera to object distance, as $\alpha$ is chosen as the inverse of the object's average depth. The paraperspective camera is a first order approximation of the perspective camera. An affine camera is paraperspective if $\mathtt{A} = \alpha[\mathtt{I}\ \ \mathbf{d}]\mathtt{R}$ with $\alpha \in \mathbb{R}^+$, $\mathbf{d} \in \mathbb{R}^{2\times 1}$ and $\bar{\mathtt{R}} \in \mathbb{SS}_{2\times 3}$. The vector $\mathbf{d}$ is related to a direction of projection. Indeed, a paraperspective camera follows the weak-perspective camera, except that the projection to the retina is along the direction defined by the perspective projection of the object's centre of mass. This allows the paraperspective camera to take the full 3D coordinates of the object's location into account in the projection process.

The orthographic, weak-perspective and paraperspective cameras form a hierarchy of approximations with gradually increasing precision to perspective projection. The affine camera models are simpler than the perspective camera because their projection process is essentially linear. However, any metric affine camera model must satisfy a set of nonlinear constraints. For these problems such as Structure-from-Motion where the camera parameters must be estimated (Poelman and Kanade, 1997; Quan, 1996; Tomasi and Kanade, 1992; Weinshall and Tomasi, 1995), this has motivated the use of approaches estimating first the general affine model (which, being non-metric, is not subject to nonlinear constraints), and then fitting the chosen metric affine camera model in a second step called metric affine correction.

## 2.3 The Singular Value Decomposition in $\mathbb{R}^{2\times 3}$

**Definition.** Our algebraic procedures are extensively based on the SVD (Golub and van Loan, 1989) in $\mathbb{R}^{2\times 3}$. An SVD $\mathtt{P} = \mathtt{U}\Sigma\mathtt{V}^\top$ must satisfy $\mathtt{U} \in \mathbb{O}_2$, $\mathtt{V} \in \mathbb{O}_3$ and $\Sigma = [\bar{\Sigma}\ \ \mathbf{0}] \in \mathbb{R}^{2\times 3}$ with $\bar{\Sigma} = \operatorname{diag}(\sigma_1, \sigma_2)$. We have that $\sigma_1, \sigma_2 \in \mathbb{R}^+$ are the singular values of $\mathtt{P}$ in decreasing order. Though the singular values are unique, an SVD is never unique. *Understanding the ambiguities of the SVD in* $\mathbb{R}^{2\times 3}$ *is of utmost importance in* *studying the ambiguities of metric affine correction.* The existence of SVD ambiguities means that for a given *reference* SVD $\mathtt{P} = \mathtt{U}\Sigma\mathtt{V}^\top$ there exist *concurrent* SVDs $\mathtt{P} = \mathtt{U}'\Sigma'\mathtt{V}'^\top$ that produce the same $\mathtt{P}$. Because $\mathtt{U}, \mathtt{U}' \in \mathbb{O}_2$ there exist $\mathtt{A} \in \mathbb{O}_2$ such that $\mathtt{U}' = \mathtt{U}\mathtt{A}$. Similarly, there exist $\mathtt{B} \in \mathbb{O}_3$ such that $\mathtt{V}' = \mathtt{V}\mathtt{B}$. Therefore the concurrent SVDs $\mathtt{U}'\Sigma'\mathtt{V}'^\top = \mathtt{U}\mathtt{A}\Sigma'\mathtt{B}^\top\mathtt{V}^\top$ and so $\Sigma = \mathtt{A}\Sigma'\mathtt{B}^\top$. Because the singular values are unique, $\Sigma$ and $\Sigma'$ must be equal up to re-ordering. We model this re-ordering with $\Sigma' = \mathtt{E}^\top\Sigma\,\mathrm{diag}(\mathtt{E}, 1)$ for some $\mathtt{E} \in \mathbb{P}_2$. By combining these two equations, we obtain $\Sigma = \mathtt{M}^\top\Sigma\mathtt{N}$ with $\mathtt{M} \stackrel{\text{def}}{=} \mathtt{A}\mathtt{E}^\top$ and $\mathtt{N} \stackrel{\text{def}}{=} \mathtt{B}\,\mathrm{diag}(\mathtt{E}, 1)^\top$. Therefore a full characterization of the SVD ambiguities is given by the family of matrices $\mathtt{A} \in \mathbb{O}_2$ and $\mathtt{B} \in \mathbb{O}_3$ such that:

$$\begin{cases} \mathtt{A} &= \mathtt{M}\mathtt{E} \\ \mathtt{B} &= \mathtt{N}\,\mathrm{diag}(\mathtt{E}, 1) \end{cases} \quad \text{with} \quad \begin{cases} \mathtt{E} \in \mathbb{P}_2,\ \mathtt{M} \in \mathbb{O}_2,\ \mathtt{N} \in \mathbb{O}_3 \\ \mathtt{M}\Sigma = \Sigma\mathtt{N}. \end{cases} \tag{1}$$

Note that the family of matrices $\mathtt{M} \in \mathbb{O}_2$ and $\mathtt{N} \in \mathbb{O}_3$ characterizes the ambiguities of the SVD up to re-ordering. The singular values of matrix $\mathtt{P}$ provide a neat way to form categories of ambiguities of the SVD. In $\mathbb{R}^{2\times3}$ this leads to four cases, which we analyze in detail below and summarize the results in table 2. In practice, this means that our algebraic procedures, which are based on taking the SVD of $\mathtt{P}$, will have to be tested against a reference SVD and four types of concurrent SVDs for ambiguities.

| | $\mathrm{rank}(\mathtt{P}) = 2$ | | $\mathrm{rank}(\mathtt{P}) = 1$ | $\mathrm{rank}(\mathtt{P}) = 0$ |
|---|---|---|---|---|
| | Case 1: $\sigma_1 > \sigma_2 > 0$ | Case 2: $\sigma_1 = \sigma_2 > 0$ | Case 3: $\sigma_1 > \sigma_2 = 0$ | Case 4: $\sigma_1 = \sigma_2 = 0$ |
| Parameters | $s_1, s_2, s_3 \in \{-1, 1\}$ $\mathtt{E} \in \mathbb{P}_2$ | $s \in \{-1, 1\}$, $\mathtt{C} \in \mathbb{O}_2$ | $s_1, s_2 \in \{-1, 1\}$ $\mathtt{C} \in \mathbb{O}_2$, $\mathtt{E} \in \mathbb{P}_2$ | $\mathtt{A} \in \mathbb{O}_2$, $\mathtt{B} \in \mathbb{O}_3$ |
| $\mathtt{A} \in \mathbb{O}_2$ | $\mathrm{diag}(s_1, s_2)\mathtt{E}$ | $\mathtt{C}$ | $\mathrm{diag}(s_1, s_2)\mathtt{E}$ | $\mathbb{O}_2$ |
| $\mathtt{B} \in \mathbb{O}_3$ | $\mathrm{diag}(s_1, s_2, s_3)\,\mathrm{diag}(\mathtt{E}, 1)$ | $\mathrm{diag}(\mathtt{C}, s)$ | $\mathrm{diag}(s_1, \mathtt{C})\,\mathrm{diag}(\mathtt{E}, 1)$ | $\mathbb{O}_3$ |
| Space | $\mathbb{B}^4$ | $\mathbb{O}_2 \times \mathbb{B}$ | $\mathbb{O}_2 \times \mathbb{B}^3$ | $\mathbb{O}_2 \times \mathbb{O}_3$ |
| $\Sigma' = \mathtt{A}^\top\Sigma\mathtt{B}$ | $= \mathtt{E}\Sigma\,\mathrm{diag}(\mathtt{E}, 1)$ | $= \Sigma = \sigma\mathtt{I}$ $\sigma \stackrel{\text{def}}{=} \sigma_1 = \sigma_2$ | $= \mathtt{E}\Sigma\,\mathrm{diag}(\mathtt{E}, 1)$ | $= \Sigma = 0$ |
| $\sigma'_1$ | $= e\sigma_1 + (1-e)\sigma_2$ | $= \sigma' = \sigma$ | $= e\sigma_1$ | $= 0$ |
| $\sigma'_2$ | $= e\sigma_2 + (1-e)\sigma_1$ | $= \sigma' = \sigma$ | $= (1-e)\sigma_1$ | $= 0$ |

Table 2: **Ambiguities of the SVD in $\mathbb{R}^{2\times3}$.** A concurrent SVD $\mathtt{U}'\Sigma'\mathtt{V}'^\top$ is related to the reference SVD $\mathtt{U}\Sigma\mathtt{V}^\top$ by $\mathtt{U}' = \mathtt{U}\mathtt{A}$, $\Sigma' = \mathtt{A}^\top\Sigma\mathtt{B}$ and $\mathtt{V}' = \mathtt{V}\mathtt{B}$. The last row of the table indicates the ambiguity space. For instance in case 1 there are no continuous ambiguities but 4 binary ones, namely 3 sign flips and 1 re-ordering. The indicator variable $e \in \{0, 1\}$ parameterizes $\mathtt{E} = e\mathtt{I} + (1-e)\tilde{\mathtt{I}}$.

**Case 1: $\sigma_1 > \sigma_2 > 0$.** This case is the most general one and has only generic ambiguities. We partition matrix $\mathtt{N} \in \mathbb{O}_3$ as:

$$\mathtt{N} = \begin{bmatrix} \bar{\mathtt{N}} & \mathbf{n}_1 \\ \mathbf{n}_2^\top & n \end{bmatrix} \qquad \text{with} \qquad \bar{\mathtt{N}} \in \mathbb{SS}_{2\times2},\ \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{SS}_{2\times1},\ n \in \mathbb{SS}_{1\times1},$$

and using $\mathtt{M}\Sigma = \Sigma\mathtt{N}$ from equation (1) we obtain:

$$\begin{bmatrix} \mathtt{M}\bar{\Sigma} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \bar{\Sigma}\bar{\mathtt{N}} & \bar{\Sigma}\mathbf{n}_1 \end{bmatrix}. \tag{2}$$

This directly gives $\mathbf{n}_1 = \mathbf{0}$. Because $\mathtt{N} \in \mathbb{O}_3$, this implies $n \in \{-1, 1\}$ and $\mathbf{n}_2 = \mathbf{0}$, and finally $\bar{\mathtt{N}} \in \mathbb{O}_2$. By parameterizing $\mathtt{M} \in \mathbb{O}_2$ and $\bar{\mathtt{N}} \in \mathbb{O}_2$ as:

$$\mathtt{M} = \begin{bmatrix} a\cos\theta & -\sin\theta \\ a\sin\theta & \cos\theta \end{bmatrix} \qquad \text{and} \qquad \bar{\mathtt{N}} = \begin{bmatrix} b\cos\mu & -\sin\mu \\ b\sin\mu & \cos\mu \end{bmatrix},$$

with $a, b \in \{-1, 1\}$ and $\theta, \mu \in \mathbb{R}$, and using the leading part $\mathtt{M}\bar{\Sigma} = \bar{\Sigma}\bar{\mathtt{N}}$ of equation (2) we obtain the following equation:

$$\begin{bmatrix} \sigma_1 a \cos\theta & -\sigma_2 \sin\theta \\ \sigma_1 a \sin\theta & \sigma_2 \cos\theta \end{bmatrix} = \begin{bmatrix} \sigma_1 b \cos\mu & -\sigma_1 \sin\mu \\ \sigma_2 b \sin\mu & \sigma_2 \cos\mu \end{bmatrix}.$$

It is trivial that element $(2, 2)$ implies $\cos\theta = \cos\mu$ and that then element $(1, 1)$ implies $a = b$. Elements $(1, 2)$ and $(2, 1)$ imply $\sin\theta = \sin\mu = 0$ and so $\theta = k\pi$, $\mu = k'\pi$, for $k, k' \in \mathbb{N}$. Finally, element $(2, 2)$ implies $k = k'$, and we thus obtain:

$$\mathtt{M} = \operatorname{diag}(s_1, s_2) \quad \text{and} \quad \mathtt{N} = \operatorname{diag}(s_1, s_2, s_3) \quad \text{with} \quad s_1, s_2, s_3 \in \{-1, 1\}.$$

We obtain a generic SVD ambiguity: sign flips on corresponding columns of $\mathtt{U}$ and $\mathtt{V}$. We finally arrive at:

$$\mathtt{A} = \operatorname{diag}(s_1, s_2)\mathtt{E} \quad \text{and} \quad \mathtt{B} = \operatorname{diag}(s_1, s_2, s_3)\operatorname{diag}(\mathtt{E}, 1) \quad \text{with} \quad s_1, s_2, s_3 \in \{-1, 1\},\ \mathtt{E} \in \mathbb{P}_2.$$

The ambiguity space is thus $\mathbb{B}^4$ (it has 4 binary possible choices). Starting from $\Sigma = \mathtt{A}\Sigma'\mathtt{B}^\top$, and replacing $\mathtt{A}$ and $\mathtt{B}$ by their expressions, we obtain $\Sigma' = \mathtt{E}\operatorname{diag}(s_1, s_2)\Sigma\operatorname{diag}(s_1, s_2, s_3)\operatorname{diag}(\mathtt{E}, 1) = \mathtt{E}\Sigma\operatorname{diag}(\mathtt{E}, 1)$.

**Case 2: $\sigma_1 = \sigma_2 > 0$.** Let $\sigma \overset{\text{def}}{=} \sigma_1 = \sigma_2$. This case is known as a 'degenerate SVD' in the literature. The SVD can be rewritten as $\mathtt{P} = \sigma\mathtt{U}[\mathtt{I}\ \mathbf{0}]\mathtt{V}^\top$ and equation (1) reduces to $\mathtt{M}[\mathtt{I}\ \mathbf{0}] = [\mathtt{I}\ \mathbf{0}]\mathtt{N}$. By partitioning $\mathtt{N} \in \mathbb{O}_3$ as we did in case 1, we obtain:

$$\begin{bmatrix} \mathtt{M} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \bar{\mathtt{N}} & \mathbf{n}_1 \end{bmatrix},$$

from which we conclude, again as in case 1, that $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{0}$, $n \in \{-1, 1\}$ and $\bar{\mathtt{N}} \in \mathbb{O}_2$. The leading part of this equation gives us $\mathtt{M} = \bar{\mathtt{N}} = \mathtt{C} \in \mathbb{O}_2$, and leads to:

$$\mathtt{M} = \mathtt{C} \quad \text{and} \quad \mathtt{N} = \operatorname{diag}(\mathtt{C}, s) \quad \text{with} \quad s \in \{-1, 1\},\ \mathtt{C} \in \mathbb{O}_2.$$

Because $\mathbb{P}_2 \subset \mathbb{O}_2$, we finally arrive at:

$$\mathtt{A} = \mathtt{C} \quad \text{and} \quad \mathtt{B} = \operatorname{diag}(\mathtt{C}, s) \quad \text{with} \quad s \in \{-1, 1\},\ \mathtt{C} \in \mathbb{O}_2.$$

The ambiguity space is thus $\mathbb{O}_2 \times \mathbb{B}$ (it has 1 continuous and 2 binary possible choices). Starting from $\Sigma = \mathtt{A}\Sigma'\mathtt{B}^\top$, and replacing $\mathtt{A}$ and $\mathtt{B}$ by their expressions, we obtain $\Sigma' = \mathtt{C}^\top\Sigma\operatorname{diag}(\mathtt{C}, s) = \Sigma = \sigma\mathtt{I}$.

**Case 3: $\sigma_1 > \sigma_2 = 0$.** In this case, we have $\bar{\Sigma} = \operatorname{diag}(\sigma_1, 0)$. We use a different partition of $\mathtt{N} \in \mathbb{O}_3$ compared to case 1 and case 2:

$$\mathtt{N} = \begin{bmatrix} n & \mathbf{n}_2^\top \\ \mathbf{n}_1 & \bar{\mathtt{N}} \end{bmatrix} \quad \text{with} \quad \bar{\mathtt{N}} \in \mathbb{SS}_{2\times 2},\ \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{SS}_{2\times 1},\ n \in \mathbb{SS}_{1\times 1}.$$

Equation (1) is then rewritten as:

$$\begin{bmatrix} \sigma_1 m_{1,1} & \mathbf{0}^\top \\ \sigma_1 m_{2,1} & \mathbf{0}^\top \end{bmatrix} = \begin{bmatrix} \sigma_1 n & \sigma_1 \mathbf{n}_2^\top \\ 0 & \mathbf{0}^\top \end{bmatrix}.$$

We thus obtain $\mathbf{n}_2 = \mathbf{0}$, and using the same reasoning as in case 1 and case 2, this implies $n \in \{-1, 1\}$, $\mathbf{n}_1 = \mathbf{0}$ and $\bar{\mathtt{N}} \in \mathbb{O}_2$. We also trivially obtain $m_{2,1} = 0$ and $m_{1,1} = n$. Because $\mathtt{M} \in \mathbb{O}_2$, this implies $m_{1,2} = 0$ and $m_{2,2} \in \{-1, 1\}$, leading to:

$$\mathtt{M} = \operatorname{diag}(s_1, s_2) \quad \text{and} \quad \mathtt{N} = \operatorname{diag}(s_1, \mathtt{C}) \quad \text{with} \quad s_1, s_2 \in \{-1, 1\},\ \mathtt{C} \in \mathbb{O}_2.$$

We finally arrive at:

$$\mathtt{A} = \operatorname{diag}(s_1, s_2)\mathtt{E}, \quad \text{and} \quad \mathtt{B} = \operatorname{diag}(s_1, \mathtt{C})\operatorname{diag}(\mathtt{E}, 1) \quad \text{with} \quad s_1, s_2 \in \{-1, 1\}, \mathtt{C} \in \mathbb{O}_2,\ \mathtt{E} \in \mathbb{P}_2.$$

The ambiguity space is thus $\mathbb{O}_2 \times \mathbb{B}^3$ (it has 1 continuous and 4 binary possible choices). Starting from $\Sigma = \mathtt{A}\Sigma'\mathtt{B}^\top$, and replacing $\mathtt{A}$ and $\mathtt{B}$ by their expressions, we obtain $\Sigma' = \mathtt{E}\operatorname{diag}(s_1, s_2)\Sigma\operatorname{diag}(s_1, \mathtt{C})\operatorname{diag}(\mathtt{E}, 1) = \mathtt{E}\Sigma\operatorname{diag}(\mathtt{E}, 1)$.

**Case 4:** $\sigma_1 = \sigma_2 = 0$**.** In this case, $\mathtt{P} = \mathtt{0} = \mathtt{U0V}^\top$ with $\mathtt{0} \in \mathbb{R}^{2\times 3}$ the all-zero matrix. We can trivially conclude that $\mathtt{M} \in \mathbb{O}_2$ and $\mathtt{N} \in \mathbb{O}_3$. In other words, $\Sigma = \mathtt{0}$ contains all the information of the SVD, and $\mathtt{U}$ and $\mathtt{V}$ are uninformative. Because $\mathbb{P}_2 \subset \mathbb{O}_2$, we arrive at:

$$\mathtt{A} \in \mathbb{O}_2 \qquad \text{and} \qquad \mathtt{B} \in \mathbb{O}_3.$$

The ambiguity space is thus $\mathbb{O}_2 \times \mathbb{O}_3$ (it has 4 continuous and 2 binary possible choices). Starting from $\Sigma = \mathtt{A}\Sigma'\mathtt{B}^\top$ we obtain $\Sigma' = \Sigma = \mathtt{0}$.

## 2.4 Methodology

For each metric affine camera, we first formulate metric affine correction using the Frobenius norm. We use this norm since we do not have specific prior information on the error distribution in $\mathtt{P}$. The Frobenius norm has been used for many similar problems in 3D computer vision. One of the most well-known examples is probably the rank-correction step of the 8 point algorithm for fundamental matrix estimation (Hartley and Zisserman, 2003). We prove the existence of a solution to this formulation, and give an algebraic procedure to compute this solution using reference SVDs. We then use the algebraic procedure to analyze the problem's generic degeneracies, corresponding to ambiguous cases which cannot be solved by any algorithm. Because the SVD is the basis of our algebraic procedures, we use its ambiguities as a basis to derive and classify the problem's generic degeneracies. We consider each of the above four cases of SVD ambiguities for each SVD involved in the algebraic procedure. This is done by substituting the reference SVD by concurrent SVDs and comparing the discrepancy in the result of the algebraic procedure. In other words, we answer the question of which ambiguities of the SVD carry over to the correction procedure. The equivalence between the problem's generic ambiguities and the algebraic procedure's can be established since in most cases the latter is non-degenerate.[3] For all cases, we show how the solution space may be entirely generated.

# 3 Orthographic Affine Correction

Defining $\bar{\mathtt{R}} \in \mathbb{S}\mathbb{S}_{2\times 3}$ as the first two rows of $\mathtt{R} \in \mathbb{S}\mathbb{O}_3$, the problem of orthographic affine correction is formulated as:

$$\min_{\mathtt{R}\in\mathbb{S}\mathbb{O}_3} \mathcal{O}_{\mathrm{OR}}(\mathtt{R}) \qquad \text{with} \qquad \mathcal{O}_{\mathrm{OR}}(\mathtt{R}) \overset{\text{def}}{=} \|\mathtt{P} - \bar{\mathtt{R}}\|_{\mathcal{F}}^2. \tag{3}$$

In solving this problem, we use an SVD of matrix $\mathtt{P} \in \mathbb{R}^{2\times 3}$. We write $\mathtt{P} = \mathtt{U}\Sigma\mathtt{V}^\top$ as the reference SVD, and $\mathtt{P} = \mathtt{U}'\Sigma'\mathtt{V}'^\top$ as the concurrent SVDs. Our algebraic procedure is given in table 3.

| |
|---|
| Function **ORAC**($\mathtt{P} \in \mathbb{R}^{2\times 3}$) |
| • Set $(\mathtt{U}, \Sigma, \mathtt{V}) \leftarrow \mathrm{SVD}(\mathtt{P})$ |
| • Set $\mathtt{R} \leftarrow \mathrm{diag}(\mathtt{U}, \det(\mathtt{U})\det(\mathtt{V}))\mathtt{V}^\top$ |
| Output $\mathtt{R} \in \mathbb{S}\mathbb{O}_3$ |

Table 3: **Algebraic procedure solving orthographic affine correction.** For $\mathrm{rank}(\mathtt{P}) = 2$ the returned solution is the problem's unique solution. For $\mathrm{rank}(\mathtt{P}) = 1$ and $\mathrm{rank}(\mathtt{P}) = 0$ the returned solution lies in the problem's solution space.

[3] Degeneracies only occur for a few special cases of paraperspective metric correction. For these cases, we specifically analyze the problem's generic degeneracies. An algebraic procedure is said to degenerate when it fails to return a valid solution. This here happens due to division by zero.

### 3.1 Solution Existence

To establish the existence of a solution we only use the reference SVD of matrix $\mathtt{P}$, that we plug into the cost function $\mathcal{O}_{\mathrm{OR}}$, giving:

$$\mathcal{O}_{\mathrm{OR}}(\mathtt{R}) \quad = \quad \|\mathtt{U}\Sigma\mathtt{V}^\top - \bar{\mathtt{R}}\|_{\mathcal{F}}^2 \quad = \quad \|\Sigma - \mathtt{U}^\top\bar{\mathtt{R}}\mathtt{V}\|_{\mathcal{F}}^2,$$

since multiplying by a unitary matrix preserves the Frobenius norm. Introducing $\bar{\mathtt{Z}} \stackrel{\text{def}}{=} \mathtt{U}^\top\bar{\mathtt{R}}\mathtt{V}$, $\bar{\mathtt{Z}} \in \mathbb{SS}_{2\times 3}$, the problem can be rewritten as:

$$\min_{\bar{\mathtt{Z}}\in\mathbb{SS}_{2\times 3}} \mathcal{O}'_{\mathrm{OR}}(\bar{\mathtt{Z}}) \qquad \text{with} \qquad \mathcal{O}'_{\mathrm{OR}}(\bar{\mathtt{Z}}) \quad = \quad \|\Sigma - \bar{\mathtt{Z}}\|_{\mathcal{F}}^2.$$

Because $\Sigma$ is a diagonal matrix, and using $\|\bar{\mathtt{Z}}\|_{\mathcal{F}}^2 = \|\bar{\mathtt{R}}\|_{\mathcal{F}}^2 = 2$, the cost function $\mathcal{O}'_{\mathrm{OR}}$ can be expanded as:

$$\mathcal{O}'_{\mathrm{OR}}(\bar{\mathtt{Z}}) \quad = \quad \|\Sigma\|_{\mathcal{F}}^2 + \|\bar{\mathtt{Z}}\|_{\mathcal{F}}^2 - 2\mathrm{tr}\left(\Sigma\bar{\mathtt{Z}}\right) \quad = \quad \sigma_1^2 + \sigma_2^2 + 2 - 2(\sigma_1 Z_{1,1} + \sigma_2 Z_{2,2}).$$

Recall that $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ and because $\bar{\mathtt{Z}} \in \mathbb{SS}_{2\times 3}$, $|Z_{1,1}| \leq 1$ and $|Z_{2,2}| \leq 1$. The cost is thus clearly minimized by choosing $Z_{1,1} = Z_{2,2} = 1$. Because $\bar{\mathtt{Z}} \in \mathbb{SS}_{2\times 3}$, we obtain $\bar{\mathtt{Z}} = [\mathtt{I}\ \mathbf{0}]$. We finally arrive at $\bar{\mathtt{R}} = \mathtt{U}[\mathtt{I}\ \mathbf{0}]\mathtt{V}^\top$ and construct the third row of $\mathtt{R}$ as the cross-product of the first two, giving:

$$\mathtt{R} \quad = \quad \mathrm{diag}(\mathtt{U}, \det(\mathtt{U})\det(\mathtt{V}))\mathtt{V}^\top.$$

A geometric interpretation of the recovered orthographic camera may be given by rewriting the result as $\bar{\mathtt{R}} = (\det(\mathtt{V})\mathtt{U})[\mathtt{I}\ \mathbf{0}]\left(\det(\mathtt{V})\mathtt{V}^\top\right)$. This shows that $\bar{\mathtt{R}}$ performs a rotation $\det(\mathtt{V})\mathtt{V}^\top$ which aligns the world to the camera's $z$-axis, followed by a parallel projection along the $z$-axis and a composed rotation and reflection $\det(\mathtt{V})\mathtt{U}$ in the image plane.

By replacing the estimated $\bar{\mathtt{Z}}$ in the cost function $\mathcal{O}'_{\mathrm{OR}}$, we obtain the value of the cost for the optimal solution as $v_{\mathrm{OR}} = \mathcal{O}'_{\mathrm{OR}}([\mathtt{I}\ \mathbf{0}])$, giving:

$$v_{\mathrm{OR}} \quad \stackrel{\text{def}}{=} \quad (\sigma_1 - 1)^2 + (\sigma_2 - 1)^2.$$

The algebraic procedure of table 3 simply follows from the derivation above. It handles all four cases of the SVD ambiguities from table 2, by returning a valid solution which always lies in the solution space, as shown in the next section.

### 3.2 Generic Problem Ambiguities

The algebraic procedure in table 3 solves formulation (3). Because the former has no degeneracies, it allows us to study the problem's generic ambiguities by analyzing the solution space. For that purpose, we use the concurrent SVDs instead of the reference SVD of $\mathtt{P}$ in our algebraic proceduce. This leads to $\mathtt{R}' = \mathrm{diag}(\mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}'))\mathtt{V}'^\top$. We then substitute $\mathtt{U}' = \mathtt{U}\mathtt{A}$ and $\mathtt{V}' = \mathtt{V}\mathtt{B}$ to analyze the problem's ambiguities for the four cases of table 2.

**Case 1:** $\sigma_1 > \sigma_2 > 0$**.** In case 1 we have $\mathtt{U}' = \mathtt{U}\,\mathrm{diag}(s_1, s_2)\mathtt{E}$ and $\mathtt{V}' = \mathtt{V}\,\mathrm{diag}(s_1, s_2, s_3)\,\mathrm{diag}(\mathtt{E}, 1)$. We thus obtain $\det(\mathtt{U}')\det(\mathtt{V}') = s_3\det(\mathtt{U})\det(\mathtt{V})$ since $\det(\mathtt{E}), s_1, s_2 \in \{-1, 1\}$. The concurrent solutions are thus:

$$\mathtt{R}' \quad = \quad \mathrm{diag}(\mathtt{U}\,\mathrm{diag}(s_1, s_2)\mathtt{E}, s_3\det(\mathtt{U})\det(\mathtt{V}))\,\mathrm{diag}(\mathtt{E}, 1)\,\mathrm{diag}(s_1, s_2, s_3)\mathtt{V}^\top.$$

Because $\mathtt{E}^2 = \mathtt{I}$ (since $\mathtt{E} \in \mathbb{P}_2 \subset \mathbb{O}_2$) and $s_1, s_2, s_3 \in \{-1, 1\}$, this simplifies to:

$$\mathtt{R}' \quad = \quad \mathrm{diag}(\mathtt{U}, \det(\mathtt{U})\det(\mathtt{V}))\mathtt{V}^\top \quad = \quad \mathtt{R}.$$

In case 1, the solution is thus unique. The optimal cost is $(\sigma_1 - 1)^2 + (\sigma_2 - 1)^2$.

**Case 2:** $\sigma_1 = \sigma_2 > 0$. In case 2 we have $\mathtt{U}' = \mathtt{UC}$ and $\mathtt{V}' = \mathtt{V}\,\mathrm{diag}(\mathtt{C}, s)$. We thus obtain $\det(\mathtt{U}')\det(\mathtt{V}') = s\det(\mathtt{U})\det(\mathtt{V})$ since $\mathtt{C} \in \mathbb{O}_2$. The concurrent solutions are thus:

$$\mathtt{R}' = \mathrm{diag}(\mathtt{UC}, s\det(\mathtt{U})\det(\mathtt{V}))\,\mathrm{diag}(\mathtt{C}, s)^\top \mathtt{V}^\top.$$

Because $\mathtt{C} \in \mathbb{O}_2$ and $s \in \{-1, 1\}$, this simplifies to:

$$\mathtt{R}' = \mathrm{diag}(\mathtt{U}, \det(\mathtt{U})\det(\mathtt{V}))\mathtt{V}^\top = \mathtt{R}.$$

In case 2, the solution is thus unique. The optimal cost can be simplified to $2(\sigma - 1)^2$, with $\sigma \stackrel{\text{def}}{=} \sigma_1 = \sigma_2$.

**Case 3:** $\sigma_1 > \sigma_2 = 0$. In case 3 we have $\mathtt{U}' = \mathtt{U}\,\mathrm{diag}(s_1, s_2)\mathtt{E}$ and $\mathtt{V}' = \mathtt{V}\,\mathrm{diag}(s_1, \mathtt{C})\,\mathrm{diag}(\mathtt{E}, 1)$. We thus obtain $\det(\mathtt{U}')\det(\mathtt{V}') = s_2\det(\mathtt{C})\det(\mathtt{U})\det(\mathtt{V})$ since $\det(\mathtt{E}), s_1 \in \{-1, 1\}$. The concurrent solutions are thus:

$$\mathtt{R}' = \mathrm{diag}(\mathtt{U}\,\mathrm{diag}(s_1, s_2)\mathtt{E}, s_2\det(\mathtt{C})\det(\mathtt{U})\det(\mathtt{V}))\,\mathrm{diag}(\mathtt{E}, 1)\,\mathrm{diag}(s_1, \mathtt{C})^\top \mathtt{V}^\top.$$

Because $\mathtt{E}^2 = \mathtt{I}$ and $s_1 \in \{-1, 1\}$, this simplifies to:

$$\mathtt{R}' = \mathrm{diag}(\mathtt{U}\,\mathrm{diag}(1, s_2), s_2\det(\mathtt{C})\det(\mathtt{U})\det(\mathtt{V}))\,\mathrm{diag}(1, \mathtt{C})^\top \mathtt{V}^\top.$$

Defining $\mathtt{G} \stackrel{\text{def}}{=} s_2\mathtt{C}\,\mathrm{diag}(1, \det(\mathtt{C}))$ with $\mathtt{G} \in \mathbb{SO}_2$ since $\mathtt{C} \in \mathbb{O}_2$ and $\det(\mathtt{G}) = 1$, we rewrite $\mathtt{R}'$ as:

$$\mathtt{R}' = \mathrm{diag}(\mathtt{U}, \det(\mathtt{U})\det(\mathtt{V}))\,\mathrm{diag}(1, \mathtt{G})^\top \mathtt{V}^\top.$$

In case 3, the solution is thus not unique: there is a rotational ambiguity in $\mathbb{SO}_2$. A geometric interpretation may be given by rewriting $\mathtt{R}'$ as the composition of three rotations $\mathtt{R}' = \mathrm{diag}(\det(\mathtt{V})\mathtt{U}, \det(\mathtt{U}))\,\mathrm{diag}(1, \mathtt{G})^\top \left(\det(\mathtt{V})\mathtt{V}^\top\right)$. This reveals that the ambiguity is a free rotation $\mathrm{diag}(1, \mathtt{G})$ around some fixed axis parallel to the image plane. This axis is the camera's $x$-axis, rotated by $\mathrm{diag}\left(\det(\mathtt{V})\mathtt{U}^\top, \det(\mathtt{U})\right)$ around the camera's $z$-axis. Choosing $\mathtt{G} = \mathtt{I}$ leads to $\mathtt{R}' = \mathtt{R}$, implying that the solution $\mathtt{R}$ returned by the algebraic procedure always lies in the solution space. The optimal cost can be simplified to $(\sigma_1 - 1)^2 + 1$.

**Case 4:** $\sigma_1 = \sigma_2 = 0$. In case 4, $\mathtt{U}' = \mathtt{UA}$ and $\mathtt{V}' = \mathtt{VB}$. The concurrent solutions are thus:

$$\mathtt{R}' = \mathrm{diag}(\mathtt{UA}, \det(\mathtt{A})\det(\mathtt{B})\det(\mathtt{U})\det(\mathtt{V}))\mathtt{B}^\top \mathtt{V}^\top.$$

Defining $\mathtt{H} \stackrel{\text{def}}{=} \mathrm{diag}(\mathtt{A}, \det(\mathtt{A})\det(\mathtt{B}))\mathtt{B}^\top$ with $\mathtt{H} \in \mathbb{SO}_3$ since $\mathtt{A} \in \mathbb{O}_2$, $\mathtt{B} \in \mathbb{O}_3$ and $\det(\mathtt{H}) = 1$, we rewrite $\mathtt{R}'$ as:

$$\mathtt{R}' = \mathrm{diag}(\mathtt{U}, \det(\mathtt{U})\det(\mathtt{V}))\mathtt{H}\mathtt{V}^\top.$$

In case 4, the solution is thus not unique: there is a rotational ambiguity in $\mathbb{SO}_3$. Geometrically, this means that any rotation solves the problem equally well. In other words, the rotation is unrecoverable. Choosing $\mathtt{H} = \mathtt{I}$ leads to $\mathtt{R}' = \mathtt{R}$, implying that the solution $\mathtt{R}$ returned by the algebraic procedure always lies in the solution space. The optimal cost can be simplified to a constant value: 2.

# 4 Weak-Perspective Affine Correction

The problem of weak-perspective affine correction is formulated as:

$$\min_{\substack{\alpha \in \mathbb{R}^+ \\ \mathtt{R} \in \mathbb{SO}_3}} \mathcal{O}_{\text{WP}}(\alpha, \mathtt{R}) \quad \text{with} \quad \mathcal{O}_{\text{WP}}(\alpha, \mathtt{R}) \stackrel{\text{def}}{=} \|\mathtt{P} - \alpha\bar{\mathtt{R}}\|_{\mathcal{F}}^2. \tag{4}$$

As in the orthographic case, we shall use the reference SVD of matrix $\mathtt{P} \in \mathbb{R}^{2\times 3}$ first and then its concurrent SVDs. Our algebraic proceduce is given in table 4.

| |
|---|
| Function **WPAC**($\mathtt{P} \in \mathbb{R}^{2\times 3}$)<br>• Set $(\mathtt{U}, \mathtt{\Sigma}, \mathtt{V}) \leftarrow \mathrm{SVD}(\mathtt{P})$, with $\mathtt{\Sigma} = [\mathrm{diag}(\sigma_1, \sigma_2)\ \ \mathbf{0}]$<br>• Set $\mathtt{R} \leftarrow \mathrm{diag}(\mathtt{U}, \det(\mathtt{U})\det(\mathtt{V}))\mathtt{V}^\top$<br>• Set $\alpha \leftarrow \frac{1}{2}(\sigma_1 + \sigma_2)$<br>Output $\alpha \in \mathbb{R}^+$ and $\mathtt{R} \in \mathbb{SO}_3$ |

Table 4: **Algebraic procedure solving weak-perspective affine correction.** For $\mathrm{rank}(\mathtt{P}) = 2$ the returned solution is the problem's unique solution. For $\mathrm{rank}(\mathtt{P}) = 1$ and $\mathrm{rank}(\mathtt{P}) = 0$ the returned solution lies in the problem's solution space.

## 4.1 Solution Existence

We follow the same steps as in the orthographic case. We first define $\bar{\mathtt{Z}} \stackrel{\text{def}}{=} \mathtt{U}^\top \bar{\mathtt{R}} \mathtt{V}$, $\bar{\mathtt{Z}} \in \mathbb{SS}_{2\times 3}$ and rewrite formulation (4) as:

$$\min_{\substack{\alpha \in \mathbb{R}^+ \\ \bar{\mathtt{Z}} \in \mathbb{SS}_{2\times 3}}} \mathcal{O}'_{\mathrm{WP}}(\alpha, \bar{\mathtt{Z}}) \qquad \text{with} \qquad \mathcal{O}'_{\mathrm{WP}}(\alpha, \bar{\mathtt{Z}}) \stackrel{\text{def}}{=} \|\mathtt{\Sigma} - \alpha \bar{\mathtt{Z}}\|^2_{\mathcal{F}}.$$

We then expand the cost function $\mathcal{O}'_{\mathrm{WP}}$, using $\|\alpha\bar{\mathtt{Z}}\|^2_{\mathcal{F}} = 2\alpha^2$, as:

$$\mathcal{O}'_{\mathrm{WP}}(\alpha, \bar{\mathtt{Z}}) \;=\; \|\mathtt{\Sigma}\|^2_{\mathcal{F}} + \|\alpha\bar{\mathtt{Z}}\|^2_{\mathcal{F}} - 2\alpha \mathrm{tr}(\mathtt{\Sigma}\bar{\mathtt{Z}}) \;=\; \sigma_1^2 + \sigma_2^2 + 2\alpha^2 - 2\alpha(\sigma_1 Z_{1,1} + \sigma_2 Z_{2,2}).$$

Since $\alpha \geq 0$ this lets us find $Z_{1,1} = Z_{2,2} = 1$, from which $\bar{\mathtt{Z}} = [\mathtt{I}\ \ \mathbf{0}]$, with the same arguments as in the orthographic case, and thus $\bar{\mathtt{R}} = \mathtt{U}[\mathtt{I}\ \ \mathbf{0}]\mathtt{V}^\top$. The cost function $\mathcal{O}'_{\mathrm{WP}}$ can then be rewritten in terms of $\alpha$ only as:

$$\mathcal{O}''_{\mathrm{WP}}(\alpha) \;\stackrel{\text{def}}{=}\; \mathcal{O}'_{\mathrm{WP}}(\alpha, [\mathtt{I}\ \ \mathbf{0}]) \;=\; \sigma_1^2 + \sigma_2^2 + 2\alpha^2 - 2\alpha(\sigma_1 + \sigma_2).$$

Differentiating with respect to $\alpha$ and nullifying leads to:

$$\frac{\partial \mathcal{O}''_{\mathrm{WP}}}{\partial \alpha}(\alpha) \;=\; 4\alpha - 2(\sigma_1 + \sigma_2) \;=\; 0,$$

and finally yields $\alpha = \frac{1}{2}(\sigma_1 + \sigma_2)$. This result ensures $\alpha \geq 0$. It is easy to verify that the estimated $\alpha$ corresponds to a minimum of the cost since $\frac{\partial^2 \mathcal{O}''_{\mathrm{WP}}}{\partial \alpha^2} = 4 > 0$. By replacing the estimated $\alpha$ in the cost function $\mathcal{O}''_{\mathrm{WP}}$, we obtain the value of the cost for the optimal solution as $\upsilon_{\mathrm{WP}} = \mathcal{O}''_{\mathrm{WP}}(\frac{1}{2}(\sigma_1 + \sigma_2))$, giving:

$$\upsilon_{\mathrm{WP}} \;\stackrel{\text{def}}{=}\; \frac{1}{2}(\sigma_1 - \sigma_2)^2.$$

The algebraic procedure of table 4 simply follows from the derivation above. It handles all four cases of the SVD ambiguities from table 2, by returning a valid solution which always lies in the solution space, as shown in the next section.

## 4.2 Generic Problem Ambiguities

The algebraic procedure in table 4 solves formulation (4). Because the former has no degeneracies, it allows us to study the problem's generic ambiguities by analyzing the solution space. As in the orthographic case, we use the concurrent SVDs of $\mathtt{P}$ in our algebraic procedure. Because the estimated rotation is obtained as in the orthographic case, the ambiguities are the same. We thus only study the ambiguities of the estimated scale.

**Case 1: $\sigma_1 > \sigma_2 > 0$.** In case 1, because the singular values are the same up to re-ordering in the reference and concurrent SVDs, we conclude that $\alpha' = \frac{1}{2}(\sigma'_1 + \sigma'_2) = \frac{1}{2}(\sigma_1 + \sigma_2) = \alpha$. In case 1, the solution is thus unique. The optimal cost is $\frac{1}{2}(\sigma_1 - \sigma_2)^2$.

**Case 2:** $\sigma_1 = \sigma_2 > 0$**.** In case 2, because the singular values are the same up to re-ordering in the reference and the concurrent SVDs, we conclude that $\alpha' = \sigma' = \sigma = \alpha$, with $\sigma \stackrel{\text{def}}{=} \sigma_1 = \sigma_2$ and $\sigma' \stackrel{\text{def}}{=} \sigma'_1 = \sigma'_2$. In case 2, the solution is thus unique. It is trivial to derive that the optimal cost vanishes; this means that the solution is exact, and that $\sigma_1 = \sigma_2 > 0$ is a sufficient constraint to characterize the family of $(2 \times 3)$ matrices representing a valid weak-perspective projection.

**Case 3:** $\sigma_1 > \sigma_2 = 0$**.** In case 3, because the singular values are the same up to re-ordering in the reference and the concurrent SVDs, we conclude that $\alpha' = \frac{1}{2}(\sigma'_1 + \sigma'_2) = \frac{1}{2}(\sigma_1 + \sigma_2) = \alpha$. In case 3, the scale is thus unique, while the estimated rotation inherits the ambiguity in $\mathbb{SO}_2$ from the orthographic case. The solution thus always lies in the solution space, as in the orthographic case. We note that the optimal scale also simplifies to $\alpha = \frac{1}{2}\sigma_1$, and the optimal cost to $\frac{1}{2}\sigma_1^2$.

**Case 4:** $\sigma_1 = \sigma_2 = 0$**.** In case 4, the scale vanishes as $\alpha' = \alpha = 0$. It is thus unique, while the rotation inherits the ambiguity in $\mathbb{SO}_3$ from the orthographic case. The solution thus always lies in the solution space, as in the orthographic case. The optimal cost vanishes.

# 5 Paraperspective Affine Correction

For a given matrix $\mathtt{D} \in \mathbb{R}^{2\times 3}$, which encapsulates the direction of projection $[\mathbf{d}^\top \ \ -1]^\top \in \mathbb{R}^{3\times 1}$ as $\mathtt{D} \stackrel{\text{def}}{=} [\mathtt{I} \ \ \mathbf{d}]$, the problem of paraperspective affine correction is stated as:

$$\min_{\substack{\alpha \in \mathbb{R}^+ \\ \mathtt{R} \in \mathbb{SO}_3}} \mathcal{O}_{\text{PP}}(\alpha, \mathtt{R}) \qquad \text{with} \qquad \mathcal{O}_{\text{PP}}(\alpha, \mathtt{R}) \ \stackrel{\text{def}}{=} \ \|\mathtt{P} - \alpha \mathtt{D}\mathtt{R}\|_{\mathcal{F}}^2. \tag{5}$$

We shall here first consider $\mathtt{D}$ as a general $(2 \times 3)$ matrix. This more general problem may be called scaled-orthonormal Procrustes analysis from 2 points, as explained below. As in the orthographic and weak-perspective cases, we use the reference SVD $\mathtt{P} = \mathtt{U}_p\Sigma_p\mathtt{V}_p^\top$ in our derivation, but also the reference SVD $\mathtt{D} = \mathtt{U}_d\Sigma_d\mathtt{V}_d^\top$. We normalize each of them to $\mathtt{P} = \tilde{\mathtt{U}}_p\Sigma_p\tilde{\mathtt{V}}_p^\top$ and $\mathtt{D} = \tilde{\mathtt{U}}_d\Sigma_d\tilde{\mathtt{V}}_d^\top$ so that $\tilde{\mathtt{V}}_p, \tilde{\mathtt{V}}_d \in \mathbb{SO}_3$ by setting:

$$\tilde{\mathtt{U}}_p \ \stackrel{\text{def}}{=} \ \det(\mathtt{V}_p)\mathtt{U}_p, \quad \tilde{\mathtt{V}}_p \ \stackrel{\text{def}}{=} \ \det(\mathtt{V}_p)\mathtt{V}_p, \quad \tilde{\mathtt{U}}_d \ \stackrel{\text{def}}{=} \ \det(\mathtt{V}_d)\mathtt{U}_d \qquad \text{and} \qquad \tilde{\mathtt{V}}_d \ \stackrel{\text{def}}{=} \ \det(\mathtt{V}_d)\mathtt{V}_d.$$

The reference and concurrent SVDs of both $\mathtt{P}$ and $\mathtt{D}$ shall be considered to study the problem's ambiguities, while only the reference SVDs are considered directly below for etablishing the solution's existence. Our algebraic procedure is given in table 5.

By interpreting each of the two rows of $\mathtt{P}$ and $\mathtt{D}$ as a pair of corresponding 3D points, problem (5) can be readily interpreted as a special case of orthonormal Procrustes analysis. In this special case, there is no translation and only two data points (whose source coordinates are defined by matrix $\mathtt{D}$ and thus in practice have the special form $[1 \ 0 \ d_1]^\top$ and $[0 \ 1 \ d_2]^\top$). This cannot be solved by existing orthonormal Procrustes analysis methods: a significant part of them does not estimate scale (Eggert et al., 1997; Kabsch, 1978; Schönemann, 1966) and methods which solve for scale either distribute it over both point sets to ensure its independence to the rotation (Horn et al., 1988) or only apply to non-coplanar points (and so to more than three points), as for instance (Schönemann and Carroll, 1970). Finally, none of the existing Procrustes analysis methods works for only two points. In paraperspective factorization for Structure-from-Motion (Poelman and Kanade, 1997), paraperspective affine correction was solved approximately. The procedure finds a suboptimal initial solution which is then refined by means of iterative nonlinear minimization. It is not clear what cost function the initial solution minimizes. In contrast, our procedure directly finds the Frobenius-norm optimal solution.

Function $\mathbf{PPAC}(\mathtt{P} \in \mathbb{R}^{2\times 3}, \mathtt{D} \in \mathbb{R}^{2\times 3})$

- Set $(\mathtt{U}_p, \Sigma_p, \mathtt{V}_p) \leftarrow \text{SVD}(\mathtt{P})$, with $\Sigma_p = [\text{diag}(\sigma_{p,1}, \sigma_{p,2}) \;\; \mathbf{0}]$
- Set $(\mathtt{U}_d, \Sigma_d, \mathtt{V}_d) \leftarrow \text{SVD}(\mathtt{D})$, with $\Sigma_d = [\text{diag}(\sigma_{d,1}, \sigma_{d,2}) \;\; \mathbf{0}]$
- Set $\mathtt{U} \leftarrow \mathtt{U}_d^\top \mathtt{U}_p$
- Set $\beta \leftarrow \sigma_{d,1}\sigma_{p,1} + \sigma_{d,2}\sigma_{p,2}$
- Set $\gamma \leftarrow \sigma_{d,1}\sigma_{p,2} + \sigma_{d,2}\sigma_{p,1}$
- Set $\delta \leftarrow \sigma_{d,1}^2 + \sigma_{d,2}^2$
- Set $\eta \leftarrow \sqrt{\beta^2 u_{2,2}^2 + \gamma^2 u_{1,2}^2}$
- Set $\alpha \leftarrow \frac{\eta}{\delta}$
- Set $\mathtt{Z} \leftarrow \text{diag}\left(\frac{1}{\eta}\,[\beta, \gamma]_\times \odot \mathtt{U}, \det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right)$
- Set $\mathtt{R} \leftarrow \mathtt{V}_d \mathtt{Z} \mathtt{V}_p^\top$

Output $\alpha \in \mathbb{R}^+$ and $\mathtt{R} \in \mathbb{SO}_3$

Table 5: **Algebraic procedure solving paraperspective affine correction.** The paraperspective case implies $\text{rank}(\mathtt{D}) = 2$. For $\text{rank}(\mathtt{P}) = 2$ the returned solution is the problem's unique solution. For $\text{rank}(\mathtt{P}) = 1$ and $\text{rank}(\mathtt{P}) = 0$ the returned solution lies in the problem's solution space, though in the latter case the procedure degenerates and fails to compute the undetermined rotation. The more general, non-paraperspective case, includes $\text{rank}(\mathtt{D}) < 2$. For $\text{rank}(\mathtt{D}) = 1$ the returned solution lies in the problem's solution space. For $\text{rank}(\mathtt{D}) = 0$ the algebraic procedure degenerates and fails to compute the undetermined scale and rotation. Both degeneracies $\text{rank}(\mathtt{P}) = 0$ and $\text{rank}(\mathtt{D}) = 0$ are discussed in §5.4 and would be easily handled as special cases.

## 5.1 Solution Existence

### 5.1.1 General Case: $\mathtt{D} \in \mathbb{R}^{2\times 3}$

Plugging the normalized reference SVD of $\mathtt{P}$ and $\mathtt{D}$ into the cost function $\mathcal{O}_{\text{PP}}$ gives:

$$\mathcal{O}_{\text{PP}}(\alpha, \mathtt{R}) \;=\; \left\|\tilde{\mathtt{U}}_p\Sigma_p\tilde{\mathtt{V}}_p^\top - \alpha\tilde{\mathtt{U}}_d\Sigma_d\tilde{\mathtt{V}}_d^\top\mathtt{R}\right\|_{\mathcal{F}}^2 \;=\; \left\|\tilde{\mathtt{U}}_d^\top\tilde{\mathtt{U}}_p\Sigma_p - \alpha\Sigma_d\tilde{\mathtt{V}}_d^\top\mathtt{R}\tilde{\mathtt{V}}_p\right\|_{\mathcal{F}}^2.$$

Introducing $\mathtt{K} \in \mathbb{R}^{2\times 2}$, $\mathtt{K} \stackrel{\text{def}}{=} \tilde{\mathtt{U}}_d^\top\tilde{\mathtt{U}}_p\bar{\Sigma}_p$ and $\tilde{\mathtt{Z}} \in \mathbb{SO}_3$, $\bar{\mathtt{Z}} \in \mathbb{SS}_{2\times 3}$ (with $\bar{\mathtt{Z}}$ the upper $(2\times 3)$ block of $\tilde{\mathtt{Z}}$), $\tilde{\mathtt{Z}} \stackrel{\text{def}}{=} \tilde{\mathtt{V}}_d^\top\mathtt{R}\tilde{\mathtt{V}}_p$, the problem can be reformulated as:

$$\min_{\substack{\alpha\in\mathbb{R}^+\\ \bar{\mathtt{Z}}\in\mathbb{SS}_{2\times 3}}} \mathcal{O}'_{\text{PP}}(\alpha, \bar{\mathtt{Z}}) \qquad \text{with} \qquad \mathcal{O}'_{\text{PP}}(\alpha, \bar{\mathtt{Z}}) \stackrel{\text{def}}{=} \|[\mathtt{K}\;\; \mathbf{0}] - \alpha\bar{\Sigma}_d\bar{\mathtt{Z}}\|_{\mathcal{F}}^2. \tag{6}$$

We expand the cost function $\mathcal{O}'_{\text{PP}}$ as:

$$\mathcal{O}'_{\text{PP}}(\alpha, \bar{\mathtt{Z}}) \;=\; \|\mathtt{K}\|_{\mathcal{F}}^2 + \alpha^2\|\bar{\Sigma}_d\bar{\mathtt{Z}}\|_{\mathcal{F}}^2 - 2\alpha\operatorname{tr}\left([\mathtt{K}\;\; \mathbf{0}]\bar{\mathtt{Z}}^\top\bar{\Sigma}_d^\top\right).$$

The first term $\|\mathtt{K}\|_{\mathcal{F}}^2 = \sigma_{p,1}^2 + \sigma_{p,2}^2$ is non minimizable. The second term $\alpha^2\|\bar{\Sigma}_d\bar{\mathtt{Z}}\|_{\mathcal{F}}^2 = \alpha^2(\sigma_{d,1}^2 + \sigma_{d,2}^2)$ is minimizable but is independent of $\bar{\mathtt{Z}}$. Because $\alpha \geq 0$, the problem can be reduced to:

$$\max_{\bar{\mathtt{Z}}\in\mathbb{SS}_{2\times 3}} \mathcal{O}''_{\text{PP}}(\bar{\mathtt{Z}}) \qquad \text{with} \qquad \mathcal{O}''_{\text{PP}}(\bar{\mathtt{Z}}) \stackrel{\text{def}}{=} \operatorname{tr}\left([\mathtt{K}\;\; \mathbf{0}]\bar{\mathtt{Z}}^\top\bar{\Sigma}_d\right).$$

Let $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^{3\times 1}$ be the first and second rows of $[\mathtt{K}\;\; \mathbf{0}]$, and let $\boldsymbol{\lambda} \in \mathbb{R}^{3\times 1}$ be a vector of Lagrange multipliers, designed to enforce the three constraints $\bar{\mathtt{Z}}\bar{\mathtt{Z}}^\top = \mathtt{I}$ from $\bar{\mathtt{Z}} \in \mathbb{SS}_{2\times 3}$. Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^{3\times 1}$ be the first and second rows of $\bar{\mathtt{Z}}$. The cost function $\mathcal{O}''_{\text{PP}}$ can be expanded to the following Lagrangian:

$$\mathcal{L}(\bar{\mathtt{Z}}, \boldsymbol{\lambda}) \;\stackrel{\text{def}}{=}\; \sigma_{d,1}\mathbf{s}_1^\top\mathbf{z}_1 + \sigma_{d,2}\mathbf{s}_2^\top\mathbf{z}_2 + \lambda_1(\|\mathbf{z}_1\|_2^2 - 1) + \lambda_2(\|\mathbf{z}_2\|_2^2 - 1) + \lambda_3\mathbf{z}_1^\top\mathbf{z}_2.$$

Differentiating with respect to $\mathbf{z}_1$ and $\mathbf{z}_2$, and nullifying gives:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\mathbf{z}_1}(\bar{\mathtt{Z}}, \boldsymbol{\lambda}) &= \sigma_{d,1}\mathbf{s}_1 + 2\lambda_1\mathbf{z}_1 + \lambda_3\mathbf{z}_2 = \mathbf{0}\\ \frac{\partial\mathcal{L}}{\partial\mathbf{z}_2}(\bar{\mathtt{Z}}, \boldsymbol{\lambda}) &= \sigma_{d,2}\mathbf{s}_2 + 2\lambda_2\mathbf{z}_2 + \lambda_3\mathbf{z}_1 = \mathbf{0}.\end{aligned}$$

This is equivalent to the following linear system:

$$\begin{bmatrix}2\lambda_1\mathtt{I} & \lambda_3\mathtt{I}\\ \lambda_3\mathtt{I} & 2\lambda_2\mathtt{I}\end{bmatrix}\begin{bmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{bmatrix} = -\begin{bmatrix}\sigma_{d,1}\mathbf{s}_1\\ \sigma_{d,2}\mathbf{s}_2\end{bmatrix}.$$

It cannot be used at this stage to solve for $\bar{\mathtt{Z}}$ since the value of $\boldsymbol{\lambda}$ is unknown, and would require one to reintroduce the three nonlinear orthonormality constraints on $\bar{\mathtt{Z}}$ to be able to find a solution. However, the above linear system allows us to express the rows of $\bar{\mathtt{Z}}$ in terms of $\mathbf{s}_1$ and $\mathbf{s}_2$ as:

$$\begin{bmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{bmatrix} = -\frac{1}{4\lambda_1\lambda_2 - \lambda_3^2}\begin{bmatrix}2\lambda_2\mathtt{I} & -\lambda_3\mathtt{I}\\ -\lambda_3\mathtt{I} & 2\lambda_1\mathtt{I}\end{bmatrix}\begin{bmatrix}\sigma_{d,1}\mathbf{s}_1\\ \sigma_{d,2}\mathbf{s}_2\end{bmatrix}.$$

By definition, the last element of $\mathbf{s}_1$ and $\mathbf{s}_2$ is zero, and because of the block structure of the above linear relationship, we conclude that the last element of $\mathbf{z}_1$ and $\mathbf{z}_2$ must be zero too. In other words, the last column of $\bar{\mathtt{Z}}$ must be zero, and we can write $\bar{\mathtt{Z}} = [\mathtt{Y}\;\; \mathbf{0}]$, for some $\mathtt{Y} \in \mathbb{O}_2$. The problem formulation (6) can thus be rewritten as:

$$\min_{\substack{\alpha\in\mathbb{R}^+\\ \mathtt{Y}\in\mathbb{O}_2}} \mathcal{O}'''_{\text{PP}}(\alpha, \mathtt{Y}) \qquad \text{with} \qquad \mathcal{O}'''_{\text{PP}}(\alpha, \mathtt{Y}) \stackrel{\text{def}}{=} \|[\mathtt{K}\;\; \mathbf{0}] - \alpha\bar{\Sigma}_d[\mathtt{Y}\;\; \mathbf{0}]\|_{\mathcal{F}}^2.$$

The cost function $\mathcal{O}'''_{\text{PP}}$ can obviously be directly rewritten as:

$$\mathcal{O}'''_{\text{PP}}(\alpha, \mathtt{Y}) \;=\; \|\mathtt{K} - \alpha\bar{\Sigma}_d\mathtt{Y}\|^2_{\mathcal{F}}.$$

We define $\mathtt{X} \stackrel{\text{def}}{=} \alpha\mathtt{Y}$. Because $\mathtt{Y} \in \mathbb{O}_2$, $\mathtt{X}$ is thus a scaled orthonormal matrix, which, using $s \in \{-1, 1\}$, $u, v \in \mathbb{R}$, can be parameterized as:

$$\mathtt{X} \;=\; \begin{bmatrix} su & v \\ -sv & u \end{bmatrix}, \tag{7}$$

with:

$$\alpha \;=\; \sqrt{u^2 + v^2}. \tag{8}$$

We have that $s = 1$ for $\mathtt{Y} \in \mathbb{SO}_2$, and $s = -1$ for $\mathtt{Y} \in \mathbb{O}_2 \backslash \mathbb{SO}_2$. The problem can thus be reformulated as:

$$\min_{\substack{u,v\in\mathbb{R}\\ s\in\{-1,1\}}} \mathcal{O}''''_{\text{PP}}(u, v, s) \qquad \text{with} \qquad \mathcal{O}''''_{\text{PP}}(u, v, s) \;\stackrel{\text{def}}{=}\; \|\mathtt{K} - \bar{\Sigma}_d\mathtt{X}\|^2_{\mathcal{F}}.$$

We first proceed to solve for $u, v$, and then determine $s$. We rewrite the cost using the column-wise vectorization operator vect as:

$$\mathcal{O}''''_{\text{PP}}(u, v, s) \;=\; \|\text{vect}(\mathtt{K}) - \text{vect}(\bar{\Sigma}_d\mathtt{X})\|^2_2 \;=\; \left\| \text{vect}(\mathtt{K}) - \mathtt{A}\begin{bmatrix} u \\ v \end{bmatrix} \right\|^2_2,$$

where $\text{vect}(\mathtt{K}) \stackrel{\text{def}}{=} [K_{1,1} \;\; K_{2,1} \;\; K_{1,2} \;\; K_{2,2}]^\top$ and $\mathtt{A} \in \mathbb{R}^{4\times 2}$ depends on $s$ and is defined as:

$$\mathtt{A} \;\stackrel{\text{def}}{=}\; \begin{bmatrix} \sigma_{d,1}s & 0 \\ 0 & -\sigma_{d,2}s \\ 0 & \sigma_{d,1} \\ \sigma_{d,2} & 0 \end{bmatrix}.$$

The solution to this linear least squares problem can be obtained as:

$$\begin{bmatrix} u \\ v \end{bmatrix} \;=\; \left(\mathtt{A}^\top\mathtt{A}\right)^{-1}\mathtt{A}^\top\text{vect}(\mathtt{K}) \;=\; \frac{1}{\sigma^2_{d,1} + \sigma^2_{d,2}}\mathtt{A}^\top\text{vect}(\mathtt{K}),$$

where we used $s \in \{-1, 1\}$ to cancel factor $s^2$. We finally obtain $u$ and $v$ as:

$$u \;=\; \frac{s\sigma_{d,1}K_{1,1} + \sigma_{d,2}K_{2,2}}{\sigma^2_{d,1} + \sigma^2_{d,2}} \tag{9}$$

$$v \;=\; \frac{\sigma_{d,1}K_{1,2} - s\sigma_{d,2}K_{2,1}}{\sigma^2_{d,1} + \sigma^2_{d,2}}. \tag{10}$$

The final step to complete this solution is to determine $s$. The problem can be formulated by plugging the estimated $u, v$ in the cost function $\mathcal{O}''''_{\text{PP}}$:

$$\min_{s\in\{-1,1\}} \mathcal{O}'''''_{\text{PP}}(s) \qquad \text{with} \qquad \mathcal{O}'''''_{\text{PP}}(s) \;\stackrel{\text{def}}{=}\; \left\|\text{vect}(\mathtt{K}) - \hat{\mathtt{A}}\text{vect}(\mathtt{K})\right\|^2_2 \;=\; \left\|(\hat{\mathtt{A}} - \mathtt{I})\text{vect}(\mathtt{K})\right\|^2_2,$$

where $\hat{\mathtt{A}} \stackrel{\text{def}}{=} \mathtt{A}\left(\mathtt{A}^\top\mathtt{A}\right)^{-1}\mathtt{A}^\top$. Matrix $\hat{\mathtt{A}} - \mathtt{I} \in \mathbb{R}^{4\times 4}$ can be expanded as:

$$\hat{\mathtt{A}} - \mathtt{I} \;=\; \frac{1}{\sigma^2_{d,1} + \sigma^2_{d,2}} \begin{bmatrix} -\sigma^2_{d,2} & 0 & 0 & s\sigma_{d,1}\sigma_{d,2} \\ 0 & -\sigma^2_{d,1} & -s\sigma_{d,1}\sigma_{d,2} & 0 \\ 0 & -s\sigma_{d,1}\sigma_{d,2} & -\sigma^2_{d,2} & 0 \\ s\sigma_{d,1}\sigma_{d,2} & 0 & 0 & -\sigma^2_{d,1} \end{bmatrix}.$$

By plugging this expansion in the cost function $\mathcal{O}''''''_{\mathrm{PP}}$, as well as the definition of $\mathtt{K} = \tilde{\mathtt{U}}\bar{\Sigma}_p$ with $\tilde{\mathtt{U}} = \tilde{\mathtt{U}}_d^\top \tilde{\mathtt{U}}_p = \left[\begin{smallmatrix} l\cos\theta & \sin\theta \\ -l\sin\theta & \cos\theta \end{smallmatrix}\right] \in \mathbb{O}_2$, we obtain the following rewriting:

$$\mathcal{O}''''''_{\mathrm{PP}}(s) \;=\; \left\| (\hat{\mathtt{A}} - \mathtt{I})\mathrm{vect}\left( \begin{bmatrix} l\cos\theta & \sin\theta \\ -l\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{p,1} & \\ & \sigma_{p,2} \end{bmatrix} \right) \right\|_2^2 .$$

Expanding, we obtain:

$$\mathcal{O}''''''_{\mathrm{PP}}(s) \;=\; \frac{1}{\left(\sigma_{d,1}^2 + \sigma_{d,2}^2\right)^2} \left\| \begin{bmatrix} \sigma_{d,2}(l\sigma_{d,2}\sigma_{p,1} - s\sigma_{d,1}\sigma_{p,2})\cos\theta) \\ \sigma_{d,1}(l\sigma_{d,1}\sigma_{p,1} - s\sigma_{d,2}\sigma_{p,2})\sin\theta) \\ \sigma_{d,2}(ls\sigma_{d,1}\sigma_{p,1} - \sigma_{d,2}\sigma_{p,2})\sin\theta) \\ \sigma_{d,1}(ls\sigma_{d,2}\sigma_{p,1} - \sigma_{d,1}\sigma_{p,2})\cos\theta) \end{bmatrix} \right\|_2^2 .$$

The value of $s$ which minimizes the absolute value of each element of the above vector is $s = l$. This may be shown by inspecting each element in turn. For the first one, for instance, ignoring the factors $\sigma_{d,2}$ and $\cos\theta$, one is left with $l\sigma_{d,2}\sigma_{p,1} - s\sigma_{d,1}\sigma_{p,2}$. Because $l, s \in \{-1, 1\}$ and $\sigma_{d,1}, \sigma_{d,2}, \sigma_{p,1}, \sigma_{p,2} \geq 0$, it is clear that $s = l$ always minimizes the value of the difference. The same reasoning holds for the four elements involved. Replacing the solution $s = l$ in the cost $\mathcal{O}''''''_{\mathrm{PP}}$, we obtain the value of the cost for the optimal solution as $v_{\mathrm{PP}} = \mathcal{O}''''''_{\mathrm{PP}}(l)$, giving:

$$v_{\mathrm{PP}} \;=\; \frac{1}{\sigma_{d,1}^2 + \sigma_{d,2}^2} \left\| \begin{bmatrix} (\sigma_{d,2}\sigma_{p,1} - \sigma_{d,1}\sigma_{p,2})\cos\theta \\ (\sigma_{d,1}\sigma_{p,1} - \sigma_{d,2}\sigma_{p,2})\sin\theta \end{bmatrix} \right\| .$$

This can be simply rewritten as:

$$v_{\mathrm{PP}} \;=\; \frac{1}{\sigma_{d,1}^2 + \sigma_{d,2}^2} \left( (\sigma_{d,2}\sigma_{p,1} - \sigma_{d,1}\sigma_{p,2})^2 u_{2,2}^2 + (\sigma_{d,1}\sigma_{p,1} - \sigma_{d,2}\sigma_{p,2})^2 u_{1,2}^2 \right) .$$

### 5.1.2 Paraperspective Case: $\mathtt{D} = [\mathtt{I}\;\; \mathbf{d}] \in \mathbb{R}^{2\times 3}$, $\mathbf{d} \in \mathbb{R}^{2\times 1}$

The paraperspective case is merely a special case where $\mathtt{D} = [\mathtt{I}\;\; \mathbf{d}]$. The main consequence of this special case is that only case 1 and case 2 in the ambiguities of the SVD of $\mathtt{D}$ apply, since $\mathrm{rank}([\mathtt{I}\;\; \mathbf{d}]) = 2$. This can be easily shown from the (reference) SVD $\mathtt{D} = \mathtt{U}_d\Sigma_d\mathtt{V}_d^\top$. By multiplying each side of this equation by its transpose, we obtain $\mathtt{I} + \mathbf{d}\mathbf{d}^\top = \mathtt{U}_d\bar{\Sigma}_d^2\mathtt{U}_d^\top$. Let $\mathbf{d} = [d_1\;\; d_2]^\top$ and $\mathbf{d}^\perp \stackrel{\mathrm{def}}{=} [-d_2\;\; d_1]^\top$. We can verify that $\frac{1}{\|\mathbf{d}\|_2}\mathbf{d}$ and $\frac{1}{\|\mathbf{d}\|_2}\mathbf{d}^\perp$ are eigenvectors of $\mathtt{I} + \mathbf{d}\mathbf{d}^\top$ associated respectively to the eigenvalues $1 + \|\mathbf{d}\|_2$ and 1. The eigenvectors can be directly used as the left singular vector of $\mathtt{I} + \mathbf{d}\mathbf{d}^\top$, while the square root of the eigenvalues gives us the singular values, leading to:

$$\mathtt{U}_d \;=\; \frac{1}{\|\mathbf{d}\|_2}\begin{bmatrix}\mathbf{d} & \mathbf{d}^\perp\end{bmatrix} \qquad \text{and} \qquad \Sigma_d \;=\; \begin{bmatrix}\mathrm{diag}(\sqrt{1 + \|\mathbf{d}\|_2^2}, 1) & \mathbf{0}\end{bmatrix}.$$

Matrix $\mathtt{V}_d$ can be easily worked out from the SVD equation. This closed-form solution is not directly useful in numerical calculations in practice since it only holds for $\mathbf{d} \neq \mathbf{0}$ and is unstable for smaller $\|\mathbf{d}\|_2$. However, it tells us a few useful facts. First, it tells us $\mathrm{rank}(\mathtt{D}) = 2$ if $\mathtt{D} = [\mathtt{I}\;\; \mathbf{d}]$ and for any $\mathbf{d} \in \mathbb{R}^{2\times 1}$. Second, it tells us that case 1 of SVD ambiguities occurs for $\mathbf{d} \neq \mathbf{0}$, since $\sigma_{d,1} = \sqrt{1 + \|\mathbf{d}\|_2^2} > \sigma_{d,2} = 1 > 0 \Leftrightarrow \mathbf{d} \neq 0$, and case 2 occurs for $\mathbf{d} = \mathbf{0}$, since and $\sigma_{d,1} = \sigma_{d,2} = 1 > 0 \Leftrightarrow \mathbf{d} = \mathbf{0}$.

## 5.2 Algebraic Procedure

There are several possible ways to turn our theory into an effective algebraic procedure. Our goal is to find the procedure with the least possible computational cost. We chose to proceed in six steps, (S1) to (S6). (S1) is to compute the SVD of $\mathtt{P}$ and $\mathtt{D}$. (S2) is to form $\mathtt{U} = \mathtt{U}_d^\top\mathtt{U}_p$. We note that $\mathtt{U} = \det(\mathtt{V}_d)\det(\mathtt{V}_p)\tilde{\mathtt{U}}$. (S3) is to compute $\beta, \gamma, \delta \in \mathbb{R}$ as:

$$\beta \;\stackrel{\mathrm{def}}{=}\; \sigma_{d,1}\sigma_{p,1} + \sigma_{d,2}\sigma_{p,2}, \quad \gamma \;\stackrel{\mathrm{def}}{=}\; \sigma_{d,1}\sigma_{p,2} + \sigma_{d,2}\sigma_{p,1} \qquad \text{and} \qquad \delta \;\stackrel{\mathrm{def}}{=}\; \sigma_{d,1}^2 + \sigma_{d,2}^2. \tag{11}$$

These three new variables are related to $u$ and $v$, as follows. By replacing $s = l$ and the entries of $\mathtt{K} = \tilde{\mathtt{U}}\bar{\Sigma}_p$ in equations (9) and (10), and using the fact that $\tilde{\mathtt{U}} \in \mathbb{O}_2$, we obtain:

$$u \;=\; \frac{\beta}{\delta}\tilde{u}_{2,2} \qquad \text{and} \qquad v \;=\; \frac{\gamma}{\delta}\tilde{u}_{1,2}. \tag{12}$$

(S4) is to compute $\eta$ and $\alpha$. The former is a new variable related to $\beta$, $\gamma$ and $\mathtt{U}$ as:

$$\eta \;\overset{\text{def}}{=}\; \sqrt{\beta^2 u_{2,2}^2 + \gamma^2 u_{1,2}^2}. \tag{13}$$

By substituting $u$ and $v$ from equations (12) in equation (8) and noting that $\tilde{u}_{1,2}^2 = u_{1,2}^2$ and $\tilde{u}_{2,2}^2 = u_{2,2}^2$, we then obtain:

$$\alpha \;=\; \frac{\eta}{\delta}. \tag{14}$$

(S5) is to compute $\mathtt{Z} = \det(\mathtt{V}_d)\det(\mathtt{V}_p)\tilde{\mathtt{Z}}$. From equation (7), and since $s = l$, we obtain by substituting $u$ and $v$ from equation (12):

$$\mathtt{X} \;=\; \det(\mathtt{V}_d)\det(\mathtt{V}_p)\frac{1}{\delta}\begin{bmatrix}\beta\, u_{1,1} & \gamma\, u_{1,2} \\ \gamma\, u_{2,1} & \beta\, u_{2,2}\end{bmatrix} \;=\; \det(\mathtt{V}_d)\det(\mathtt{V}_p)\frac{1}{\delta}\,[\beta,\gamma]_\times \odot \mathtt{U}. \tag{15}$$

By definition, we have $\mathtt{Y} = \frac{1}{\alpha}\mathtt{X}$ and $\bar{\mathtt{Z}} = [\frac{1}{\alpha}\mathtt{X}\ \mathbf{0}]$. Replacing $\mathtt{X}$ by its expression (15), and forming the third row of $\tilde{\mathtt{Z}} \in \mathbb{SO}_3$ as the cross-product of its first two rows, we obtain:

$$\mathtt{Z} \;=\; \det(\mathtt{V}_d)\det(\mathtt{V}_p)\tilde{\mathtt{Z}} \;=\; \begin{bmatrix} \frac{1}{\eta}\,[\beta,\gamma]_\times \odot \mathtt{U} & \begin{matrix}0\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & \det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\end{bmatrix}. \tag{16}$$

(S6) is to compute $\mathtt{R}$. By definition, it simply comes as $\mathtt{R} = \tilde{\mathtt{V}}_d\tilde{\mathtt{Z}}\tilde{\mathtt{V}}_p^\top = \det(\mathtt{V}_d)\mathtt{V}_d\det(\mathtt{V}_d)\det(\mathtt{V}_p)\mathtt{Z}\det(\mathtt{V}_p)\mathtt{V}_p^\top = \mathtt{V}_d\mathtt{Z}\mathtt{V}_p^\top$. Our algebraic procedure is summarized in table 5.

### 5.3 Closed-Form Solution

We may obtain a closed-form solution for the scale and rotation from the SVD of $\mathtt{P}$ and $\mathtt{D}$. By substituting the expressions for $\beta$, $\gamma$ and $\eta$ from equations (11) and (13) into equations (14) and (16), we arrive at:

$$\begin{aligned}
\alpha \;&=\; \frac{\sqrt{(\sigma_{d,1}\sigma_{p,1}+\sigma_{d,2}\sigma_{p,2})^2 u_{2,2}^2 + (\sigma_{d,1}\sigma_{p,2}+\sigma_{d,2}\sigma_{p,1})^2 u_{1,2}^2}}{\sigma_{d,1}^2+\sigma_{d,2}^2} \\
\mathtt{R} \;&=\; \mathtt{V}_d\,\mathrm{diag}\left(\frac{[\sigma_{d,1}\sigma_{p,1}+\sigma_{d,2}\sigma_{p,2}, \sigma_{d,1}\sigma_{p,2}+\sigma_{d,2}\sigma_{p,1}]_\times \odot \mathtt{U}}{\sqrt{(\sigma_{d,1}\sigma_{p,1}+\sigma_{d,2}\sigma_{p,2})u_{2,2} + (\sigma_{d,1}\sigma_{p,2}+\sigma_{d,2}\sigma_{p,1})u_{1,2}}}, \det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right)\mathtt{V}_p^\top.
\end{aligned}$$

Computing directly this closed-form solution is more expensive than running our algebraic procedure.

### 5.4 Degeneracies of the Algebraic Procedure

The algebraic procedure in table 5 may degenerate by not returning a valid solution. This specifically happens when $\delta$ or $\eta$ (or both) vanishes, since this would cause a division by zero. We have established that a degeneracy occurs when either $\mathrm{rank}(\mathtt{P}) = 0$ or $\mathrm{rank}(\mathtt{D}) = 0$ (or both). Both cases may be easily handled as special cases in the algebraic procedure.

**Case** $\mathrm{rank}(\mathtt{P}) = 0$ **and** $\mathrm{rank}(\mathtt{D}) > 0$**.** For $\mathrm{rank}(\mathtt{P}) = 0$, the cost function can be simplified to $\mathcal{O}_{\mathrm{PP}} = \|\alpha\mathtt{D}\mathtt{R}\|_{\mathcal{F}}^2 = \alpha^2\|\mathtt{D}\|_{\mathcal{F}}^2$, simplifying formulation (5) to $\min_{\alpha\in\mathbb{R}^+}\alpha^2$, and leaving us with the solution $\alpha = 0$ and $\mathtt{R} \in \mathbb{SO}_3$ undetermined. In this case, the algebraic procedure in table 5 computes $\beta = \gamma = 0$, and thus $\eta = 0$. This leads to the correct scale $\alpha = 0$, but fails to compute $\mathtt{Z}$ since it involves $\frac{1}{\eta}$, and thus fails to compute $\mathtt{R}$.

**Case** $\mathrm{rank}(\mathtt{D}) = 0$**.** For $\mathrm{rank}(\mathtt{D}) = 0$, the cost function reduces to the constant value $\mathcal{O}_{\text{PP}} = \|\mathtt{P}\|_{\mathcal{F}}^2$, leaving us with the solution $\alpha \in \mathbb{R}^+$ and $\mathtt{R} \in \mathbb{SO}_3$, both undetermined. In this case, the algebraic procedure in table 5 computes $\beta = \gamma = 0$, and thus $\eta = 0$, but also $\delta = 0$. This thus fails to compute $\alpha$ since it involves $\frac{1}{\delta}$, and $\mathtt{Z}$ since it involves $\frac{1}{\eta}$, and thus fails to compute $\mathtt{R}$.

## 5.5 Generic Problem Ambiguities

The algebraic procedure in table 5 solves formulation (5). Following the discussion directly above, the former degenerates only for $\mathrm{rank}(\mathtt{P}) = 0$ or $\mathrm{rank}(\mathtt{D}) = 0$. In other words, we may, as in the orthographic and weak-perspective cases, use our algebraic procedure to study the problem's generic ambiguities by analyzing the solution space, in non-degenerate cases. Our analysis proceeds by evaluating our algebraic procedure directly on the concurrent SVDs, and comparing the result with the reference SVDs. Because the SVD of matrices $\mathtt{P}, \mathtt{D} \in \mathbb{R}^{2\times 3}$ are involved in the procedure, this leaves $4 \times 4 = 16$ possible combinations of the four cases of SVD ambiguities given in table 2. We note that in the specific paraperspective case, since one has $\mathtt{D} = [\mathtt{I}\ \mathbf{0}]$ and thus $\mathrm{rank}(\mathtt{D}) = 2$, only case 1 and case 2 are left to study in the SVD of $\mathtt{D}$, leading to an overall number of $2 \times 4 = 8$ relevant cases. For completeness however we shall study the complete set of 16 cases. Our results are summarized in table 6. The detailed proof can be found in the appendix.

| P | Case 1 | | | | Case 2 | | | | Case 3 | | | | Case 4 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| D | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| Scale | ✓ | ✓ | ✓ | ✗ | ✓ | ✓ | ✓ | ✗ | ✓ | ✓ | ✓ | ✗ | ✓ | ✓ | ✓ | ✗ |
| Rotation | ✓ | ✓ | $\mathcal{A}_{\text{D}}$ | ✗ | ✓ | ✓ | $\mathcal{A}_{\text{D}}$ | ✗ | $\mathcal{A}_{\text{P}}$ | $\mathcal{A}_{\text{P}}$ | $\mathcal{A}_{\text{DP}}$ | ✗ | ✗ | ✗ | ✗ | ✗ |

Table 6: **Summary of results on the solution uniqueness in paraperspective affine correction.** This table summarizes the 16 possible cases of SVD ambiguities for the input matrices $\mathtt{P}, \mathtt{D} \in \mathbb{R}^{2\times 3}$ for the general problem. Paraperspective affine correction is only concerned with the 8 cases in grey since it implies $\mathrm{rank}(\mathtt{D}) = 2$ and thus $\mathtt{D}$ only falls in case 1 and case 2. The *Scale* and *Rotation* rows indicate respectively if the scale and rotation are uniquely recoverable with ✓, completely unrecoverable (which is equivalent to an ambiguity in $\mathbb{R}^+$ and $\mathbb{SO}_3$ respectively) with ✗ or partially recoverable, indicating the type of rotational ambiguity with $\mathcal{A}_{\text{D}} \equiv \mathbb{SO}_2$, $\mathcal{A}_{\text{P}} \equiv \mathbb{SO}_2$ or $\mathcal{A}_{\text{DP}} \equiv \mathbb{SO}_2^2$ (see main text for details).

We identified four types of rotational ambiguities: $\mathcal{A}_{\text{D}} \equiv \mathbb{SO}_2$, $\mathcal{A}_{\text{P}} \equiv \mathbb{SO}_2$, $\mathcal{A}_{\text{DP}} \equiv \mathbb{SO}_2^2$ and $\mathbb{SO}_3$. We describe their structure below, and indicate how to obtain the whole spectrum of solutions, from the last step of the algebraic procedure forming the rotation as $\mathtt{R} \leftarrow \mathtt{V}_d\mathtt{Z}\mathtt{V}_p^\top$:

- **Ambiguities of type** $\mathcal{A}_{\text{D}}$**.** This type of ambiguities manifests itself when one may choose any matrix $\mathtt{G}_d \in \mathbb{SO}_2$ to generate an admissible solution as $\mathtt{R} \leftarrow \mathtt{V}_d \,\mathrm{diag}(1, \mathtt{G}_d)\mathtt{Z}\mathtt{V}_p^\top$.
- **Ambiguities of type** $\mathcal{A}_{\text{P}}$**.** This type of ambiguities manifests itself when one may choose any matrix $\mathtt{G}_p \in \mathbb{SO}_2$ to generate an admissible solution as $\mathtt{R} \leftarrow \mathtt{V}_d\mathtt{Z}\,\mathrm{diag}(1, \mathtt{G}_p^\top)\mathtt{V}_p^\top$.
- **Ambiguities of type** $\mathcal{A}_{\text{DP}}$**.** This type of ambiguities combines types $\mathcal{A}_{\text{D}}$ and $\mathcal{A}_{\text{P}}$. It manifests itself when one may choose any matrices $\mathtt{G}_d, \mathtt{G}_p \in \mathbb{SO}_2$ to generate an admissible solution as $\mathtt{R} \leftarrow \mathtt{V}_d \,\mathrm{diag}(1, \mathtt{G}_d)\mathtt{Z}\,\mathrm{diag}(1, \mathtt{G}_p^\top)\mathtt{V}_p^\top$.
- **Ambiguities of type** $\mathbb{SO}_3$**.** This type of ambiguities means that the rotation is completely unrecoverable. Any rotation thus solves the problem equally well.

It should be noted from table 6 that in the paraperspective case, where $\mathrm{rank}(\mathtt{D}) = 2$, the procedure is guaranteed to complete and to return a valid solution, in the sense that it always lies in the solution space. The ambiguities are then simple: there are none for $\mathrm{rank}(\mathtt{P}) = 2$ (case 1 and case 2), there is one in $\mathbb{SO}_2$ for $\mathrm{rank}(\mathtt{P}) = 1$ (case 3), and the rotation is unrecoverable for $\mathrm{rank}(\mathtt{P}) = 0$ (case 4). The scale is always simple.

# 6 Empirical Evaluation and Comparison

## 6.1 Compared Methods

For each camera model, we have compared our algebraic procedure, abbreviated by Alg, to an alternative method which we have designed to assess the precision of the numerical results and stability of Alg. The alternative method is based on a quaternion parameterization and global polynomial optimization. It is abbreviated by Poly. The quaternion parameterization is important since 9 entries of a rotation matrix would be too many parameters for current global polynomial optimization toolboxes. We have also used the result of both Alg and Poly to initialize a direct iterative solution of the metric affine correction problems (3), (4) and (5) with an interior-point method from Matlab's optimization toolbox. The resulting methods are respectively abbreviated AlgRef and PolyRef.

**General methodology for global polynomial optimization correction methods.** In Poly, the rotation matrix $\mathtt{R}$ is parameterized by a unit quaternion represented by $\mathbf{a} \in \mathbb{R}^4$, $\|\mathbf{a}\|_2 = 1$. With this representation, $\mathtt{R}$ becomes a quadratic function of $\mathbf{a}$, denoted as $\mathtt{R} = \mathcal{R}(\mathbf{a})$ and defined as:

$$\mathcal{R}(\mathbf{a}) \stackrel{\text{def}}{=} \begin{bmatrix} a_1^2 + a_2^2 - a_3^2 - a_4^2 & 2a_2a_3 - 2a_1a_4 & 2a_2a_4 + 2a_1a_3 \\ 2a_2a_3 + 2a_1a_4 & a_1^2 - a_2^2 + a_3^2 - a_4^2 & 2a_3a_4 - 2a_1a_2 \\ 2a_2a_4 - 2a_1a_3 & 2a_3a_4 + 2a_1a_2 & a_1^2 - a_2^2 - a_3^2 + a_4^2 \end{bmatrix}.$$

Using the quaternion representation, each metric affine correction problem can be turned into a polynomial optimization problem with 4 unknowns. We used Matlab's symbolic toolbox to expand each polynomial cost, and used it as a cost function in the Globtipoly 3 toolbox (Henrion et al., 2009). Globtipoly does global polynomial optimization, and, for our problems, always returns two conjugate solutions representing the same geometric solution, since $\mathcal{R}(\mathbf{a}) = \mathcal{R}(-\mathbf{a})$.

**The orthographic camera.** We define $\bar{\mathcal{R}}(\mathbf{a})$ as the first two rows of $\mathcal{R}(\mathbf{a})$. Problem (3) can thus be rewritten as:

$$\min_{\substack{\mathbf{a}\in\mathbb{R}^4 \\ \|\mathbf{a}\|_2^2=1}} \mathcal{O}''_{\text{OR}}(\mathbf{a}) \qquad \text{with} \qquad \mathcal{O}''_{\text{OR}}(\mathbf{a}) \stackrel{\text{def}}{=} \|\mathtt{P} - \bar{\mathcal{R}}(\mathbf{a})\|_{\mathcal{F}}^2.$$

We have changed the original unity constraint $\|\mathbf{a}\|_2 = 1$ to the equivalent quadratic constraint $\|\mathbf{a}\|_2^2 = 1$. It is clear that $\mathcal{O}''_{\text{OR}}$ is a quartic polynomial in the entries of $\mathbf{a}$, which is solved by Globtipoly 3 under the quadratic constraint.

**The weak-perspective camera.** The product $\alpha\bar{\mathtt{R}}$ involved in problem (4) can be rewritten as:

$$\alpha\bar{\mathtt{R}} \;=\; \alpha\bar{\mathcal{R}}(\mathbf{a}) \;=\; \bar{\mathcal{R}}(\sqrt{\alpha}\mathbf{a}) \;=\; \bar{\mathcal{R}}(\mathbf{b}) \qquad \text{with} \qquad \mathbf{b} \stackrel{\text{def}}{=} \sqrt{\alpha}\mathbf{a}.$$

We have defined $\mathbf{b} \in \mathbb{R}^4$ as a general (non-unitary) quaternion. Problem (4) may thus be reformulated as:

$$\min_{\mathbf{b}\in\mathbb{R}^4} \mathcal{O}'''_{\text{WP}}(\mathbf{b}) \qquad \text{with} \qquad \mathcal{O}'''_{\text{WP}}(\mathbf{b}) \stackrel{\text{def}}{=} \|\mathtt{P} - \bar{\mathcal{R}}(\mathbf{b})\|_{\mathcal{F}}^2.$$

It is clear that $\mathcal{O}'''_{\text{WP}}$ is a quartic polynomial in the entries of $\mathbf{b}$. Once solved by Globtipoly 3, we finally extract the scale as $\alpha \leftarrow \|\mathbf{b}\|_2^2$ and build the rotation as $\mathtt{R} \leftarrow \mathcal{R}\left(\frac{\mathbf{b}}{\|\mathbf{b}\|_2}\right)$.

**The paraperspective camera.** The product $\alpha\mathtt{DR}$ involved in problem (5) can be rewritten as:

$$\alpha\mathtt{DR} \;=\; \alpha\mathtt{D}\mathcal{R}(\mathbf{a}) \;=\; \mathtt{D}\mathcal{R}(\sqrt{\alpha}\mathbf{a}) \;=\; \mathtt{D}\mathcal{R}(\mathbf{b}).$$

Problem (5) may thus be reformulated as:

$$\min_{\mathbf{b}\in\mathbb{R}^4} \mathcal{O}''''''_{\text{PP}}(\mathbf{b}) \qquad \text{with} \qquad \mathcal{O}''''''_{\text{PP}}(\mathbf{b}) \stackrel{\text{def}}{=} \|\mathtt{P} - \mathtt{D}\mathcal{R}(\mathbf{b})\|_{\mathcal{F}}^2.$$

It is clear that $\mathcal{O}''''''_{\text{PP}}$ is a quartic polynomial in the entries of $\mathbf{b}$. Once solved by Globtipoly 3, we finally extract the scale and rotation from $\mathbf{b}$ as described for the weak-perspective camera.

| | *Average discrepancy to* PolyRef | | | *Standard deviation* | | |
|---|---|---|---|---|---|---|
| | Poly | Alg | AlgRef | Poly | Alg | AlgRef |
| Rotation | $7.7 \times 10^{-5}$ | $1.7 \times 10^{-8}$ | $1.0 \times 10^{-8}$ | $3.6 \times 10^{-5}$ | $2.2 \times 10^{-7}$ | $2.2 \times 10^{-7}$ |

Table 7: **Experimental results for orthographic correction.** For each unorthographicity rate, the average and standard deviation of the discrepancy to PolyRef was computed over 10,000 trials. The left part of the table shows the average over unorthographicity of the average discrepancy, while the right part shows the average over unorthographicity of the discrepancy's standard deviation.

| | *Average discrepancy to* PolyRef | | | *Standard deviation* | | |
|---|---|---|---|---|---|---|
| | Poly | Alg | AlgRef | Poly | Alg | AlgRef |
| Scale | $2.7 \times 10^{-5}$ | $7.5 \times 10^{-9}$ | $1.1 \times 10^{-9}$ | $1.4 \times 10^{-5}$ | $5.0 \times 10^{-10}$ | $1.6 \times 10^{-9}$ |
| Rotation | $1.8 \times 10^{-5}$ | $1.4 \times 10^{-8}$ | $7.6 \times 10^{-9}$ | $1.2 \times 10^{-5}$ | $1.5 \times 10^{-7}$ | $1.5 \times 10^{-7}$ |

Table 8: **Experimental results for weak-perspective correction.** For each unweakperspectiveness rate, the average and standard deviation of the discrepancy to PolyRef was computed over 10,000 trials. The left part of the table shows the average over unweakperspectiveness of the average discrepancy, while the right part shows the average over unweakperspectiveness of the discrepancy's standard deviation.

| | *Average discrepancy to* PolyRef | | | *Standard deviation* | | |
|---|---|---|---|---|---|---|
| | Poly | Alg | AlgRef | Poly | Alg | AlgRef |
| Scale | $2.5 \times 10^{-5}$ | $7.4 \times 10^{-9}$ | $1.7 \times 10^{-9}$ | $5.4 \times 10^{-7}$ | $4.3 \times 10^{-11}$ | $1.5 \times 10^{-10}$ |
| Rotation | $1.4 \times 10^{-5}$ | $1.4 \times 10^{-8}$ | $8.0 \times 10^{-9}$ | $2.5 \times 10^{-6}$ | $4.0 \times 10^{-9}$ | $4.6 \times 10^{-9}$ |

Table 9: **Experimental results for paraperspective correction.** For each unparaperspectiveness rate, the average and standard deviation of the discrepancy to PolyRef was computed over 10,000 trials. The left part of the table shows the average over unparaperspectiveness of the average discrepancy, while the right part shows the average over unparaperspectiveness of the discrepancy's standard deviation.

## 6.2 Experimental Setup

We simulated orthographic, weak-perspective and paraperspective cameras by randomly drawing scale factors, projection directions and rotations. In order to test the correction methods, we perturbed the simulated perfect camera matrices by random additive gaussian noise with strength varying from 0 to 100% of the camera matrix' norm. The noise strength thus represents *unorthographicity*, *unweakperspectiveness* and *unparaperspectiveness* respectively. We averaged the results over 10,000 trials for each noise strength. For each camera model, we ran the four methods, and computed their correction residual, given by the value of $\sqrt{\mathcal{O}_{\text{OR}}}$, $\sqrt{\mathcal{O}_{\text{WP}}}$ and $\sqrt{\mathcal{O}_{\text{PP}}}$ from problems (3), (4) and (5) respectively, evaluated with the returned solutions. We also computed the discrepancy in rotation and scale (only for the weak-perspective and paraperspective cameras) between PolyRef and the other three methods. The reason we chose PolyRef as a reference here is because it is the one which gives the most objective evaluation of Alg, the method we ultimately want to evaluate here.

## 6.3 Experimental Results

**Numerical accuracy.** Our results are given in figure 1, 2 and 3 for the orthographic, weak-perspective and paraperspective cameras respectively. We observe similar trends for the three camera models in several respects. First, we observe that all four methods gave the same correction residual. For the orthographic and weak-perspective cameras, the residual behaves as a quadratic function of the unorthographicity and unweakperspectiveness rates respectively, while for the paraperspective camera, it behaves as a linear function of the unparaperspectiveness rate. The discrepancy between PolyRef and the other methods remains, roughly speaking, at a constant order of magnitude, and is similar for scale and rotation. The discrepancy is of an order of magnitude between $10^{-4}$ and $10^{-5}$ for Poly, $10^{-8}$ for Alg and $10^{-9}$ for AlgRef (with the exception of greater than 60% unorthographicity, unweakperspectiveness and unparaperspectiveness rates where it reaches $10^{-8}$ in rotation). To be more precise, we report the average discrepancies and standard deviations in tables 7, 8 and 9. From these results, it is clear that all methods find very similar results, corresponding to the same geometric solution. PolyRef, Alg and AlgRef all have a very good numerical precision while Poly is slightly less numerically stable.

**Computation time.** We monitored the computation time taken by each method. The following table gives the average and standard deviation over all trials, in seconds:

| | Poly | Alg | AlgRef | PolyRef |
|---|---|---|---|---|
| Orthographic camera | | | | |
| | $1.0 \times 10^{-1} \pm 9.0 \times 10^{-3}$ | $1.0 \times 10^{-4} \pm 1.2 \times 10^{-5}$ | $3.5 \times 10^{-2} \pm 8.2 \times 10^{-3}$ | $5.2 \times 10^{-2} \pm 2.6 \times 10^{-2}$ |
| Weak-perspective camera | | | | |
| | $9.5 \times 10^{-2} \pm 7.2 \times 10^{-3}$ | $1.2 \times 10^{-4} \pm 1.9 \times 10^{-5}$ | $3.6 \times 10^{-2} \pm 9.1 \times 10^{-3}$ | $5.2 \times 10^{-2} \pm 2.0 \times 10^{-2}$ |
| Paraperspective camera | | | | |
| | $2.1 \times 10^{-1} \pm 1.5 \times 10^{-2}$ | $1.7 \times 10^{-4} \pm 5.8 \times 10^{-5}$ | $4.6 \times 10^{-2} \pm 1.1 \times 10^{-2}$ | $6.2 \times 10^{-2} \pm 2.7 \times 10^{-2}$ |

As for the numerical accuracy, we can draw general conclusions valid for the three camera models. As expected, Poly is the slowest, requiring about a tenth of a second per run, followed by AlgRef and PolyRef, being an order of magnitude faster. Finally, Alg is the fastest, as it is three orders of magnitude faster than Poly.

**Conclusion.** For all three camera models, the proposed algebraic procedures Alg are numerically stable as they agree to a very good precision with an iterative interior-point method with two types of initialization: from methods Alg themself (methods AlgRef) and from methods Poly, which are methods using a global polynomial solver (methods PolyRef). Methods Alg are between three and four orders of magnitude more accurate than methods Poly. The algebraic procedures Alg are also between two and three orders of magnitude faster than all other methods.

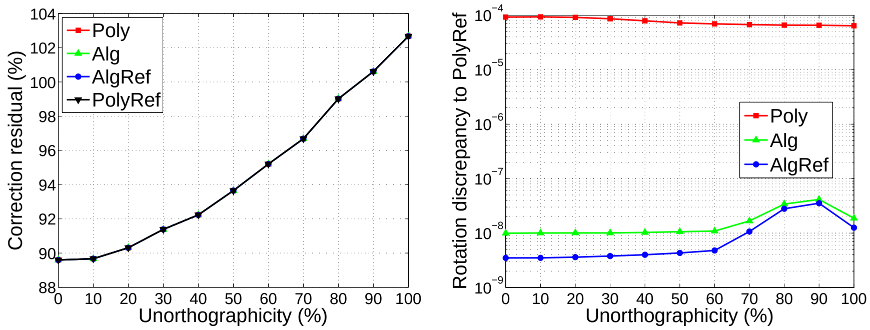


Figure 1: **Experimental results for orthographic metric correction.** The four methods are undistinguishable on the left-most graph. Note the logarithmic scale on the vertical axis of the right-most graph.

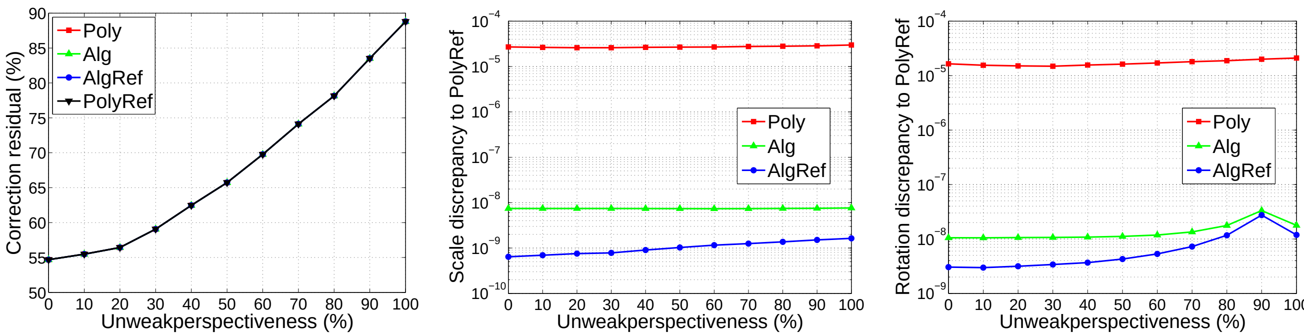


Figure 2: **Experimental results for weak-perspective metric correction.** The four methods are undistinguishable on the left-most graph. Note the logarithmic scale on the vertical axes of the middle and right-most graphs.

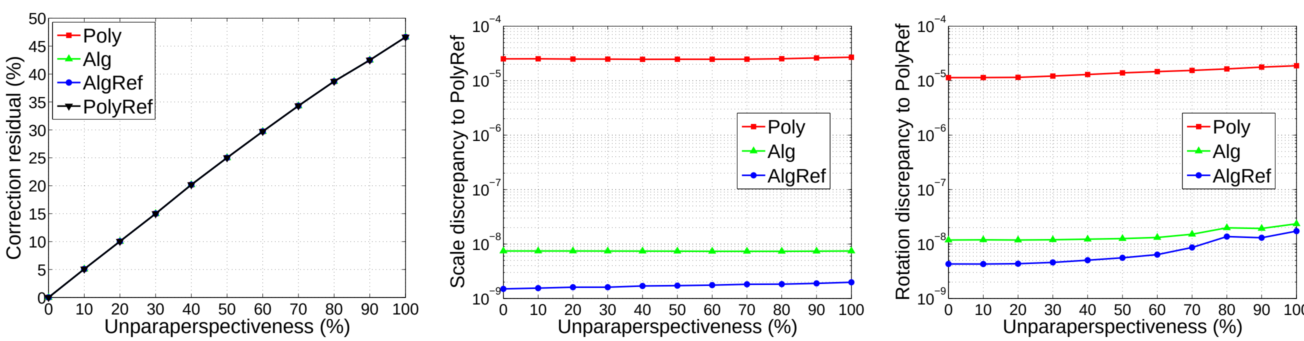


Figure 3: **Experimental results for paraperspective metric correction.** The four methods are undistinguishable on the left-most graph. Note the logarithmic scale on the vertical axes of the middle and right-most graphs.

## 7 Conclusion

We have given algebraic procedures to solve the problems of finding the closest orthographic, weak-perspective or paraperspective projection matrix to a given general affine projection represented by a $(2 \times 3)$ matrix. These algebraic procedures efficiently implement closed-form solutions which we derived, along with a complete characterization of the problems' generic ambiguities and solution spaces. We proved that the level of ambiguity can be determined from the input affine projection matrix' rank. For a full-rank input matrix the solution is always unique. For a rank-one input matrix (representing an affine projection to an image line) the rotation is 1D ambiguous and the weak-perspective and paraperspective scale is unique. For a zero matrix (representing an affine projection to a single image point) the rotation is unrecoverable and the scale vanishes. So as to obtain this complete characterization of ambiguities, we have introduced an original methodology based on the SVD in $\mathbb{R}^{2\times 3}$, which we think could benefit to other problems such as Procrustes analysis. Interesting topics of future work are the detection and handling of ambiguities in the presence of noise.

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# A Generic Problem Ambiguities in Paraperspective Correction

The algebraic procedure in table 5 solves formulation (5). The former degenerates for $\text{rank}(\mathtt{P}) = 0$ or $\text{rank}(\mathtt{D}) = 0$ (case 4 of the SVD ambiguities in table 2). We have already established the problem's specific generic ambiguities for these cases in §5.4. Our algebraic procedure thus allows us to study the problem's generic ambiguities by analyzing the solution space obtained from the concurrent SVDs in 9 cases of SVD ambiguities: cases 1, 2 and 3 for P combined with cases 1, 2 and 3 for D. This is done by analyzing the concurrent solutions to our procedure of table 5 step by step. The 5 remaining cases, corresponding to the algebraic procedure's degeneracies, are included for completeness, and indicated by an asterisk mark.

**Preliminaries.** We represent a permutation matrix $\mathtt{E} \in \mathbb{P}_2$ by a binary value $e \in \{0, 1\}$ with $\mathtt{E} = e\mathtt{I} + (1 - e)\tilde{\mathtt{I}}$. We use the indicator function $\mathbb{1} : \{\text{true}, \text{false}\} \to \{0, 1\}$, with:

$$\begin{aligned} \mathbb{1}_{e=e'} &\stackrel{\text{def}}{=} ee' + (1-e)(1-e') \\ \mathbb{1}_{e\neq e'} &\stackrel{\text{def}}{=} e(1-e') + (1-e)e'. \end{aligned}$$

We use the rules $\mathbb{1}^2_{e=e'} = \mathbb{1}_{e=e'}$, $\mathbb{1}_{e=e'}\mathbb{1}_{e\neq e'} = 0$ and $\mathbb{1}_{e=e'} + \mathbb{1}_{e\neq e'} = 1$.

**Case 1–1:** $\sigma_{p,1} > \sigma_{p,2} > 0$**,** $\sigma_{d,1} > \sigma_{d,2} > 0$**.**

- Set $(\mathtt{U}'_p, \Sigma'_p, \mathtt{V}'_p) \leftarrow \text{SVD}(\mathtt{P})$, $\Sigma'_p = [\text{diag}(\sigma'_{p,1}, \sigma'_{p,2})\ \ \mathbf{0}]$

From table 2 we have, for $\mathtt{E}_p \in \mathbb{P}_2$, $e_p \in \{0,1\}$, $s_{p,1}, s_{p,2}, s_{p,3} \in \{-1, 1\}$:

$$\begin{aligned} \mathtt{U}'_p &= \mathtt{U}_p \,\text{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p && (17)\\ \Sigma'_p &= \mathtt{E}_p\Sigma_p \,\text{diag}(\mathtt{E}_p, 1) && (18)\\ \mathtt{V}'_p &= \mathtt{V}_p \,\text{diag}(s_{p,1}, s_{p,2}, s_{p,3}) \,\text{diag}(\mathtt{E}_p, 1) && (19)\\ \mathtt{E}_p &= e_p\mathtt{I} + (1-e_p)\tilde{\mathtt{I}} && (20)\\ \sigma'_{p,1} &= e_p\sigma_{p,1} + (1-e_p)\sigma_{p,2} && (21)\\ \sigma'_{p,2} &= e_p\sigma_{p,2} + (1-e_p)\sigma_{p,1}. && (22) \end{aligned}$$

- Set $(\mathtt{U}'_d, \Sigma'_d, \mathtt{V}'_d) \leftarrow \text{SVD}(\mathtt{P})$, $\Sigma'_d = [\text{diag}(\sigma'_{d,1}, \sigma'_{d,2})\ \ \mathbf{0}]$

From table 2 we have, for $\mathtt{E}_d \in \mathbb{P}_2$, $e_d \in \{0,1\}$, $s_{d,1}, s_{d,2}, s_{d,3} \in \{-1, 1\}$:

$$\begin{aligned} \mathtt{U}'_d &= \mathtt{U}_d \,\text{diag}(s_{d,1}, s_{d,2})\mathtt{E}_d && (23)\\ \Sigma'_d &= \mathtt{E}_d\Sigma_d \,\text{diag}(\mathtt{E}_d, 1) && (24)\\ \mathtt{V}'_d &= \mathtt{V}_d \,\text{diag}(s_{d,1}, s_{d,2}, s_{d,3}) \,\text{diag}(\mathtt{E}_d, 1) && (25)\\ \mathtt{E}_d &= e_d\mathtt{I} + (1-e_d)\tilde{\mathtt{I}} && (26)\\ \sigma'_{d,1} &= e_d\sigma_{d,1} + (1-e_d)\sigma_{d,2} && (27)\\ \sigma'_{d,2} &= e_d\sigma_{d,2} + (1-e_d)\sigma_{d,1}. && (28) \end{aligned}$$

- Set $\mathtt{U}' \leftarrow \mathtt{U}_d'^\top \mathtt{U}_p'$

Using equations (17) and (23) we obtain:

$$\mathtt{U}' = (\mathtt{U}_d \operatorname{diag}(s_{d,1}, s_{d,2})\mathtt{E}_d)^\top \mathtt{U}_p \operatorname{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p = \mathtt{E}_d \operatorname{diag}(s_{d,1}, s_{d,2})\mathtt{U} \operatorname{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p. \tag{29}$$

This implies $|\mathtt{U}'| = \mathtt{E}_d|\mathtt{U}|\mathtt{E}_p$, and thus, using equations (20) and (26) we obtain:

$$|\mathtt{U}'| = (e_d\mathtt{I} + (1-e_d)\tilde{\mathtt{I}})|\mathtt{U}|(e_p\mathtt{I} + (1-e_p)\tilde{\mathtt{I}}) = e_de_p|\mathtt{U}| + e_d(1-e_p)|\mathtt{U}|\tilde{\mathtt{I}} + (1-e_d)e_p\tilde{\mathtt{I}}|\mathtt{U}| + (1-e_d)(1-e_p)\tilde{\mathtt{I}}|\mathtt{U}|\tilde{\mathtt{I}}.$$

Because $\mathtt{U} \in \mathbb{O}_2$, $|\mathtt{U}|\tilde{\mathtt{I}} = \tilde{\mathtt{I}}|\mathtt{U}|$ and $|\mathtt{U}| = \tilde{\mathtt{I}}|\mathtt{U}|\tilde{\mathtt{I}}$, and so:

$$|\mathtt{U}'| = (e_de_p + (1-e_d)(1-e_p))|\mathtt{U}| + (e_d(1-e_p) + (1-e_d)e_p)|\mathtt{U}|\tilde{\mathtt{I}} = \mathbb{1}_{e_d=e_p}|\mathtt{U}| + \mathbb{1}_{e_d\neq e_p}|\mathtt{U}|\tilde{\mathtt{I}}. \tag{30}$$

- Set $\beta' \leftarrow \sigma'_{d,1}\sigma'_{p,1} + \sigma'_{d,2}\sigma'_{p,2}$

Using equations (21), (22), (27) and (28) we obtain:

$$\begin{aligned}\beta' &= (e_d\sigma_{d,1} + (1-e_d)\sigma_{d,2})(e_p\sigma_{p,1} + (1-e_p)\sigma_{p,2}) + (e_d\sigma_{d,2} + (1-e_d)\sigma_{d,1})(e_p\sigma_{p,2} + (1-e_p)\sigma_{p,1}) \\ &= (e_de_p + (1-e_d)(1-e_p))(\sigma_{d,1}\sigma_{p,1} + \sigma_{d,2}\sigma_{p,2}) + (e_d(1-e_p) + (1-e_d)e_p)(\sigma_{d,1}\sigma_{p,2} + \sigma_{d,2}\sigma_{p,1}) \\ &= \mathbb{1}_{e_d=e_p}\beta + \mathbb{1}_{e_d\neq e_p}\gamma. \end{aligned} \tag{31}$$

- Set $\gamma' \leftarrow \sigma'_{d,1}\sigma'_{p,2} + \sigma'_{d,2}\sigma'_{p,1}$

Using equations (21), (22), (27) and (28) we obtain:

$$\begin{aligned}\gamma' &= (e_d\sigma_{d,1} + (1-e_d)\sigma_{d,2})(e_p\sigma_{p,2} + (1-e_p)\sigma_{p,1}) + (e_d\sigma_{d,2} + (1-e_d)\sigma_{d,1})(e_p\sigma_{p,1} + (1-e_p)\sigma_{p,2}) \\ &= (e_de_p + (1-e_d)(1-e_p))(\sigma_{d,1}\sigma_{p,2} + \sigma_{d,2}\sigma_{p,1}) + (e_d(1-e_p) + (1-e_d)e_p)(\sigma_{d,1}\sigma_{p,1} + \sigma_{d,2}\sigma_{p,2}) \\ &= \mathbb{1}_{e_d=e_p}\gamma + \mathbb{1}_{e_d\neq e_p}\beta. \end{aligned} \tag{32}$$

- Set $\delta' \leftarrow \sigma'^2_{d,1} + \sigma'^2_{d,2}$

Using equations (27) and (28) we obtain:

$$\begin{aligned}\delta' &= (e_d\sigma_{d,1} + (1-e_d)\sigma_{d,2})^2 + (e_d\sigma_{d,2} + (1-e_d)\sigma_{d,1})^2 \\ &= e_d\sigma^2_{d,1} + (1-e_d)\sigma^2_{d,2} + e_d\sigma^2_{d,2} + (1-e_d)\sigma^2_{d,1} \\ &= \sigma^2_{d,1} + \sigma^2_{d,2} \\ &= \delta. \end{aligned} \tag{33}$$

- Set $\eta' \leftarrow \sqrt{\beta'^2 u'^2_{2,2} + \gamma'^2 u'^2_{1,2}}$

Equation (30) implies $u'^2_{1,2} = \mathbb{1}_{e_d=e_p}u^2_{1,2} + \mathbb{1}_{e_d\neq e_p}u^2_{2,2}$ and $u'^2_{2,2} = \mathbb{1}_{e_d=e_p}u^2_{2,2} + \mathbb{1}_{e_d\neq e_p}u^2_{1,2}$, from which we may expand $\eta'$ using equations (31) and (32) as:

$$\begin{aligned}\eta' &= \sqrt{(\mathbb{1}_{e_d=e_p}\beta^2 + \mathbb{1}_{e_d\neq e_p}\gamma^2)(\mathbb{1}_{e_d=e_p}u^2_{2,2} + \mathbb{1}_{e_d\neq e_p}u^2_{1,2}) + (\mathbb{1}_{e_d=e_p}\gamma^2 + \mathbb{1}_{e_d\neq e_p}\beta^2)(\mathbb{1}_{e_d=e_p}u^2_{1,2} + \mathbb{1}_{e_d\neq e_p}u^2_{2,2})} \\ &= \sqrt{(\mathbb{1}_{e_d=e_p} + \mathbb{1}_{e_d\neq e_p})(\beta^2 u^2_{2,2} + \gamma^2 u^2_{1,2})} \\ &= \eta. \end{aligned} \tag{34}$$

- Set $\alpha' \leftarrow \frac{\eta'}{\delta'}$

Using equations (33) and (34) we obtain $\alpha' = \frac{\eta}{\delta} = \alpha$.

- Set $\mathtt{Z}' \leftarrow \operatorname{diag}\left(\frac{1}{\eta'}\left[\beta', \gamma'\right]_\times \odot \mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right)$

Equations (31) and (32) imply $[\beta', \gamma']_\times = \mathtt{E}_d\,[\beta,\gamma]_\times\,\mathtt{E}_p$ from which we may expand the leading block using equations (29) and (34) as:

$$\begin{aligned}\frac{1}{\eta'}\,[\beta',\gamma']_\times \odot \mathtt{U}' &= \frac{1}{\eta}(\mathtt{E}_d\,[\beta,\gamma]_\times\,\mathtt{E}_p) \odot (\mathtt{E}_d\,\mathrm{diag}(s_{d,1}, s_{d,2})\mathtt{U}\,\mathrm{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p) \\ &= \frac{1}{\eta}\mathtt{E}_d([\beta,\gamma]_\times \odot (\mathrm{diag}(s_{d,1}, s_{d,2})\mathtt{U}\,\mathrm{diag}(s_{p,1}, s_{p,2})))\mathtt{E}_p \qquad (35)\end{aligned}$$

From equations (19), (25) and (29) we simplify the bottom right entry to:

$$\det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p) = s_{d,3}s_{p,3}\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p). \qquad (36)$$

- Set $\mathtt{R}' \leftarrow \mathtt{V}'_d\mathtt{Z}'\mathtt{V}'^\top_p$

Using equations (19), (25), (35) and (36) we obtain:

$$\begin{aligned}\mathtt{R}' &= \mathtt{V}_d\,\mathrm{diag}(s_{d,1}, s_{d,2}, s_{d,3})\,\mathrm{diag}(\mathtt{E}_d, 1) \\ &\quad \mathrm{diag}\left(\frac{1}{\eta}\mathtt{E}_d([\beta,\gamma]_\times \odot (\mathrm{diag}(s_{d,1}, s_{d,2})\mathtt{U}\,\mathrm{diag}(s_{p,1}, s_{p,2})))\mathtt{E}_p, s_{d,3}s_{p,3}\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right) \\ &\quad \mathrm{diag}(\mathtt{E}_p, 1)\,\mathrm{diag}(s_{p,1}, s_{p,2}, s_{p,3})\mathtt{V}_p^\top \\ &= \mathtt{V}_d\,\mathrm{diag}\left(\frac{1}{\eta}([\beta,\gamma]_\times \odot (\mathrm{diag}(s^2_{d,1}, s^2_{d,2})\mathtt{U}\,\mathrm{diag}(s^2_{p,1}, s^2_{p,2}))), s^3_{d,3}s^3_{p,3}\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right)\mathtt{V}_p^\top \\ &= \mathtt{V}_d\mathtt{Z}\mathtt{V}_p^\top \\ &= \mathtt{R}.\end{aligned}$$

In case 1–1, the solution is thus unique. The optimal cost is:

$$v_{\mathrm{PP}} = \frac{1}{\sigma^2_{d,1} + \sigma^2_{d,2}}\left((\sigma_{d,2}\sigma_{p,1} - \sigma_{d,1}\sigma_{p,2})^2 u^2_{2,2} + (\sigma_{d,1}\sigma_{p,1} - \sigma_{d,2}\sigma_{p,2})^2 u^2_{1,2}\right).$$

**Case 1–2: $\sigma_{p,1} > \sigma_{p,2} > 0$, $\sigma_{d,1} = \sigma_{d,2} > 0$.** We define $\sigma_d \stackrel{\mathrm{def}}{=} \sigma_{d,1} = \sigma_{d,2}$.

- Set $(\mathtt{U}'_p, \Sigma'_p, \mathtt{V}'_p) \leftarrow \mathrm{SVD}(\mathtt{P})$, $\Sigma'_p = [\mathrm{diag}(\sigma'_{p,1}, \sigma'_{p,2})\ \ \mathbf{0}]$

See case 1–1.

- Set $(\mathtt{U}'_d, \Sigma'_d, \mathtt{V}'_d) \leftarrow \mathrm{SVD}(\mathtt{D})$, $\Sigma'_d = [\mathrm{diag}(\sigma'_{d,1}, \sigma'_{d,2})\ \ \mathbf{0}]$

From table 2 we have, for $\mathtt{C}_d \in \mathbb{O}_2$ and $s_d \in \{-1, 1\}$:

$$\begin{aligned}\mathtt{U}'_d &= \mathtt{U}_d\mathtt{C}_d &(37)\\ \Sigma'_d &= \Sigma_d &(38)\\ \mathtt{V}'_d &= \mathtt{V}_d\,\mathrm{diag}(\mathtt{C}_d, s_d) &(39)\\ \sigma'_d &\stackrel{\mathrm{def}}{=} \sigma'_{d,1} = \sigma'_{d,2} = \sigma_d. &(40)\end{aligned}$$

- Set $\mathtt{U}' \leftarrow \mathtt{U}'^\top_d\mathtt{U}'_p$

Using equations (17) and (37) we obtain:

$$\mathtt{U}' = (\mathtt{U}_d\mathtt{C}_d)^\top\mathtt{U}_p\,\mathrm{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p = \mathtt{C}_d^\top\mathtt{U}\,\mathrm{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p. \qquad (41)$$

- Set $\beta' \leftarrow \sigma'_{d,1}\sigma'_{p,1} + \sigma'_{d,2}\sigma'_{p,2}$

Using equations (21), (22) and (40) we obtain:

$$\begin{aligned}\beta' &= \sigma_d(\sigma'_{p,1} + \sigma'_{p,2}) \\ &= \sigma_d(e_p\sigma_{p,1} + (1-e_p)\sigma_{p,2} + e_p\sigma_{p,2} + (1-e_p)\sigma_{p,1}) \\ &= \sigma_d((e_p + (1-e_p))(\sigma_{p,1} + \sigma_{p,2}) \\ &= \sigma_d(\sigma_{p,1} + \sigma_{p,2}) \\ &= \beta.\end{aligned}$$

- Set $\gamma' \leftarrow \sigma'_{d,1}\sigma'_{p,2} + \sigma'_{d,2}\sigma'_{p,1}$

Using equations (21), (22) and (40) we obtain:

$$\begin{aligned}\gamma' &= \sigma_d(\sigma'_{p,2} + \sigma'_{p,1}) \\ &= \sigma_d(e_p\sigma_{p,2} + (1-e_p)\sigma_{p,1} + e_p\sigma_{p,1} + (1-e_p)\sigma_{p,2}) \\ &= \sigma_d((e_p + (1-e_p))(\sigma_{p,1} + \sigma_{p,2}) \\ &= \sigma_d(\sigma_{p,1} + \sigma_{p,2}) \\ &= \gamma.\end{aligned}$$

We note that $\gamma' = \gamma = \beta' = \beta$.

- Set $\delta' \leftarrow \sigma'^2_{d,1} + \sigma'^2_{d,2}$

Using equation (40) we obtain $\delta' = \sigma'^2_d + \sigma'^2_d = \sigma^2_d + \sigma^2_d = \sigma^2_{d,1} + \sigma^2_{d,2} = \delta$.

- Set $\eta' \leftarrow \sqrt{\beta'^2 u'^2_{2,2} + \gamma'^2 u'^2_{1,2}}$

We have seen that $\beta' = \beta = \gamma' = \gamma$. Therefore, $\eta' = \beta\sqrt{u'^2_{2,2} + u'^2_{1,2}}$. Because $\mathtt{U}' \in \mathbb{O}_2$, $\sqrt{u'^2_{2,2} + u'^2_{1,2}} = 1$ and this simplifies to $\eta' = \beta$. Because $\mathtt{U} \in \mathbb{O}_2$, we also obtain $\eta = \beta\sqrt{u^2_{2,2} + u^2_{1,2}} = \beta$, and thus $\eta' = \eta$.

- Set $\alpha' \leftarrow \frac{\eta'}{\delta'}$

We have $\alpha' = \frac{\eta}{\delta} = \alpha$ since $\eta' = \eta$ and $\delta' = \delta$.

- Set $\mathtt{Z}' \leftarrow \operatorname{diag}\left(\frac{1}{\eta'}\left[\beta', \gamma'\right]_\times \odot \mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right)$

Because $\eta' = \beta' = \gamma'$ we have $\frac{1}{\eta'}\left[\beta', \gamma'\right]_\times = \mathtt{I}$, from which, using equation (41) we simplify the leading block as:

$$\frac{1}{\eta'}\left[\beta', \gamma'\right]_\times \odot \mathtt{U}' = \mathtt{U}' = \mathtt{C}_d^\top \mathtt{U}\operatorname{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p. \tag{42}$$

Using equations (19), (39) and (41) we simplify the bottom right entry to:

$$\det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p) = s_d s_{p,3}\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p). \tag{43}$$

- Set $\mathtt{R}' \leftarrow \mathtt{V}'_d\mathtt{Z}'\mathtt{V}'^\top_p$

Using equations (19), (39), (42) and (43) we obtain:

$$\begin{aligned}\mathtt{R}' &= \mathtt{V}_d\operatorname{diag}(\mathtt{C}_d, s_d)\operatorname{diag}(\mathtt{U}', s_d s_{p,3}\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p))\operatorname{diag}(\mathtt{E}_p, 1)\operatorname{diag}(s_{p,1}, s_{p,2}, s_{p,3})\mathtt{V}_p^\top \\ &= \mathtt{V}_d\operatorname{diag}\left(\mathtt{C}_d\mathtt{C}_d^\top\mathtt{U}'\operatorname{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p\mathtt{E}_p\operatorname{diag}(s_{p,1}, s_{p,2}), s_d^2 s_{p,3}^2\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right)\mathtt{V}_p^\top \\ &= \mathtt{V}_d\mathtt{Z}\mathtt{V}_p^\top \\ &= \mathtt{R}.\end{aligned}$$

In case 1–2, the solution is thus unique. The optimal cost simplifies to $\frac{1}{2}(\sigma_{p,1} - \sigma_{p,2})^2$. This is the same as the general weak-perspective cost.

**Case 1–3: $\sigma_{p,1} > \sigma_{p,2} > 0$, $\sigma_{d,1} > \sigma_{d,2} = 0$.** In case 1–3 almost all the early steps are similar to case 1–1. In fact, they could even be further simplified owing to $\sigma_{d,2} = 0$, though this is not necessary for the demonstration. We thus here only give the steps which differ from case 1–1.

- Set $(\mathtt{U}'_d, \Sigma'_d, \mathtt{V}'_d) \leftarrow \mathrm{SVD}(\mathtt{D})$, $\Sigma'_d = [\operatorname{diag}(\sigma'_{d,1}, \sigma'_{d,2})\ \ \mathbf{0}]$

From table 2 we have, for $\mathtt{E}_d \in \mathbb{P}_2$, $e_d \in \{0,1\}$, $\mathtt{C}_d \in \mathbb{O}_2$ and $s_{d,1}, s_{d,2} \in \{-1,1\}$:

$$\mathtt{U}'_d = \mathtt{U}_d \operatorname{diag}(s_{d,1}, s_{d,2})\mathtt{E}_d \tag{44}$$
$$\Sigma'_d = \mathtt{E}_d\Sigma_d \operatorname{diag}(\mathtt{E}_d, 1) \tag{45}$$
$$\mathtt{V}'_d = \mathtt{V}_d \operatorname{diag}(s_{d,1}, \mathtt{C}_d) \operatorname{diag}(\mathtt{E}_d, 1) \tag{46}$$
$$\sigma'_{d,1} = e_d\sigma_{d,1} \tag{47}$$
$$\sigma'_{d,2} = (1-e_d)\sigma_{d,1}. \tag{48}$$

- Set $\mathtt{Z}' \leftarrow \operatorname{diag}\left(\frac{1}{\eta'}[\beta',\gamma']_\times \odot \mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right)$

The leading block is expanded as in case 1–1. From equation (46), (19) and (29) we simplify the bottom right entry to:

$$s_{d,2}s_{p,3}\det(\mathtt{C}_d)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p). \tag{49}$$

- Set $\mathtt{R}' \leftarrow \mathtt{V}'_d\mathtt{Z}'\mathtt{V}'^\top_p$

Using equations (19), (35), (46) and (49) we obtain:

$$\begin{aligned}
\mathtt{R}' &= \mathtt{V}_d \operatorname{diag}(s_{d,1}, \mathtt{C}_d)\operatorname{diag}(\mathtt{E}_d, 1) \\
&\quad \operatorname{diag}\left(\frac{1}{\eta}\mathtt{E}_d([\beta,\gamma]_\times \odot (\operatorname{diag}(s_{d,1}, s_{d,2})\mathtt{U}\operatorname{diag}(s_{p,1}, s_{p,2})))\mathtt{E}_p, s_{d,2}s_{p,3}\det(\mathtt{C}_d)\det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right) \\
&\quad \operatorname{diag}(\mathtt{E}_p, 1)\operatorname{diag}(s_{p,1}, s_{p,2}, s_{p,3})\mathtt{V}_p^\top \\
&= \mathtt{V}_d\operatorname{diag}(1,\mathtt{C}_d)\operatorname{diag}\left(\frac{1}{\eta}[\beta,\gamma]_\times \odot (\operatorname{diag}(1, s_{d,2})\mathtt{U}), s_{d,2}\det(\mathtt{C}_d)\det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right)\mathtt{V}_p^\top \\
&= \mathtt{V}_d\operatorname{diag}(1,\mathtt{G}_d)\operatorname{diag}\left(\frac{1}{\eta}[\beta,\gamma]_\times \odot \mathtt{U}, \det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right)\mathtt{V}_p^\top \\
&= \mathtt{V}_d\operatorname{diag}(1,\mathtt{G}_d)\mathtt{Z}\mathtt{V}_p^\top,
\end{aligned}$$

with $\mathtt{G}_d \in \mathbb{SO}_2$, $\mathtt{G}_d \overset{\text{def}}{=} s_{d,2}\mathtt{C}_d\operatorname{diag}(1, \det(\mathtt{C}_d))$.

In case 1–3, the scale $\alpha$ is thus unique, but the rotation $\mathtt{R}$ has an ambiguity of type $\mathcal{A}_{\text{D}} \equiv \mathbb{SO}_2$. Choosing $\mathtt{G}_d = \mathtt{I}$ leads to $\mathtt{R}' = \mathtt{R}$, implying that the solution $\mathtt{R}$ returned by the algebraic procedure always lies in the solution space. The optimal cost simplifies to $\sigma_{p,1}^2 u_{1,2}^2 + \sigma_{p,2}^2 u_{2,2}^2$.

***Case 1–4: $\sigma_{p,1} > \sigma_{p,2} > 0$, $\sigma_{d,1} = \sigma_{d,2} = 0$.** In case 1–4, $\mathtt{D} = \mathtt{0} \in \mathbb{R}^{2\times 3}$, and no information can be retrieved on the scale and the rotation, since they both vanish from the cost function, as studied in §5.4. The overall ambiguity space is thus $\mathbb{SO}_3 \times \mathbb{R}^+$. The cost is constant and given by $\sigma_{p,1}^2 + \sigma_{p,2}^2$.

**Case 2–1: $\sigma_{p,1} = \sigma_{p,2} > 0$, $\sigma_{d,1} > \sigma_{d,2} > 0$.** We define $\sigma_p \overset{\text{def}}{=} \sigma_{p,1} = \sigma_{p,2}$. Case 2–1 shares similarities with case 1–2 to which we substantially refer.

- Set $(\mathtt{U}'_p, \Sigma'_p, \mathtt{V}'_p) \leftarrow \text{SVD}(\mathtt{P})$, $\Sigma'_p = [\operatorname{diag}(\sigma'_{p,1}, \sigma'_{p,2})\ \mathbf{0}]$

From table 2 we have, for $\mathtt{C}_p \in \mathbb{O}_2$ and $s_p \in \{-1,1\}$:

$$\mathtt{U}'_p = \mathtt{U}_p\mathtt{C}_p \tag{50}$$
$$\Sigma'_p = \Sigma_p \tag{51}$$
$$\mathtt{V}'_p = \mathtt{V}_p\operatorname{diag}(\mathtt{C}_p, s_p) \tag{52}$$
$$\sigma'_p \overset{\text{def}}{=} \sigma'_{p,1} = \sigma'_{p,2} = \sigma_p. \tag{53}$$

- Set $(\mathtt{U}'_d, \Sigma'_d, \mathtt{V}'_d) \leftarrow \text{SVD}(\mathtt{D})$, $\Sigma'_d = [\operatorname{diag}(\sigma'_{d,1}, \sigma'_{d,2})\ \mathbf{0}]$

See case 1–1.

- Set $\mathtt{U}' \leftarrow \mathtt{U}_d'^{\top}\mathtt{U}_p'$

Using equations (23) and (50) we obtain:
$$\mathtt{U}' = \mathtt{E}_d \operatorname{diag}(s_{d,1}, s_{d,2})\mathtt{U}\mathtt{C}_p. \tag{54}$$

- Set $\beta' \leftarrow \sigma'_{d,1}\sigma'_{p,1} + \sigma'_{d,2}\sigma'_{p,2}$

By following the same reasoning as in case 1–2 but switching the role of $p$ and $d$, we obtain from equations (27), (28) and (53) $\beta' = \sigma'_p(\sigma'_{d,1} + \sigma'_{d,2}) = \sigma_p(\sigma_{d,1} + \sigma_{d,2}) = \beta$.

- Set $\gamma' \leftarrow \sigma'_{d,1}\sigma'_{p,2} + \sigma'_{d,2}\sigma'_{p,1}$

By following the same reasoning as in case 1–2 but switching the role of $p$ and $d$, we obtain from equations (27), (28) and (53) $\gamma' = \gamma = \beta$.

- Set $\delta' \leftarrow \sigma'^2_{d,1} + \sigma'^2_{d,2}$

See case 1–1, $\delta' = \delta$.

- Set $\eta' \leftarrow \sqrt{\beta'^2 u'^2_{2,2} + \gamma'^2 u'^2_{1,2}}$

See case 1–2, $\eta' = \beta' = \beta = \eta$.

- Set $\alpha' \leftarrow \frac{\eta'}{\delta'}$

See case 1–2, $\alpha' = \frac{\eta}{\delta} = \alpha$ since $\eta' = \eta$ and $\delta' = \delta$.

- Set $\mathtt{Z}' \leftarrow \operatorname{diag}\left(\frac{1}{\eta'}[\beta', \gamma']_\times \odot \mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}_d')\det(\mathtt{V}_p')\right)$

Following case 1–2, and using equations (25), (52) and (54), the leading block is expanded to $\mathtt{E}_d \operatorname{diag}(s_{d,1}s_{d,2})\mathtt{U}\mathtt{C}_p$ and the bottom right entry to $s_{d,3}s_p \det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)$.

- Set $\mathtt{R}' \leftarrow \mathtt{V}_d'\mathtt{Z}'\mathtt{V}_p'^{\top}$

By replacing $\mathtt{Z}'$ by its expression directly above, and using equations (25) and (52), we show with similar steps to case 1–2 that $\mathtt{R}' = \mathtt{V}_d\mathtt{Z}\mathtt{V}_p^{\top} = \mathtt{R}$.

In case 2–1 the solution is thus unique. The optimal cost simplifies to $\frac{\sigma_p^2}{\sigma_{d,1}^2+\sigma_{d,2}^2}(\sigma_{d,1} - \sigma_{d,2})^2$.

**Case 2–2: $\sigma_{p,1} = \sigma_{p,2} > 0$, $\sigma_{d,1} = \sigma_{d,2} > 0$.** We define $\sigma_p \overset{\text{def}}{=} \sigma_{p,1} = \sigma_{p,2}$ and $\sigma_d \overset{\text{def}}{=} \sigma_{d,1} = \sigma_{d,2}$.

- Set $(\mathtt{U}_p', \Sigma_p', \mathtt{V}_p') \leftarrow \text{SVD}(\mathtt{P})$, $\Sigma_p' = [\operatorname{diag}(\sigma'_{p,1}, \sigma'_{p,2})\ \ \mathbf{0}]$

See case 2–1.

- Set $(\mathtt{U}_d', \Sigma_d', \mathtt{V}_d') \leftarrow \text{SVD}(\mathtt{D})$, $\Sigma_d' = [\operatorname{diag}(\sigma'_{d,1}, \sigma'_{d,2})\ \ \mathbf{0}]$

See case 1–2.

- Set $\mathtt{U}' \leftarrow \mathtt{U}_d'^{\top}\mathtt{U}_p'$

From equations (37) and (50) we obtain:
$$\mathtt{U}' = \mathtt{C}_d^{\top}\mathtt{U}\mathtt{C}_p. \tag{55}$$

- Set $\beta' \leftarrow \sigma'_{d,1}\sigma'_{p,1} + \sigma'_{d,2}\sigma'_{p,2}$

From equations (40) and (53) we obtain $\beta' = 2\sigma'_d\sigma'_p = 2\sigma_d\sigma_p = \beta$.

- Set $\gamma' \leftarrow \sigma'_{d,1}\sigma'_{p,2} + \sigma'_{d,2}\sigma'_{p,1}$

From equations (40) and (53) we obtain $\gamma' = 2\sigma'_d\sigma'_p = 2\sigma_d\sigma_p = \gamma$. We note that $\gamma' = \gamma = \beta = \beta'$.

- Set $\delta' \leftarrow \sigma'^2_{d,1} + \sigma'^2_{d,2}$

See case 1–2, $\delta' = \delta$.

- Set $\eta' \leftarrow \sqrt{\beta'^2 u'^2_{2,2} + \gamma'^2 u'^2_{1,2}}$

| |
|---|
| See case 1–2 and case 2–1, $\eta' = \eta$ and $\eta = \gamma = \beta$. |

- Set $\alpha' \leftarrow \frac{\eta'}{\delta'}$

| |
|---|
| We have $\alpha' = \frac{\eta}{\delta} = \alpha$ since $\eta' = \eta$ and $\delta' = \delta$. |

- Set $\mathtt{Z}' \leftarrow \text{diag}\left(\frac{1}{\eta'}[\beta', \gamma']_\times \odot \mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right)$

| |
|---|
| Following case 1–2, and using equations (39), (52) and (55), the leading block is rewritten as $\mathtt{U}' = \mathtt{C}_d^\top \mathtt{U}\mathtt{C}_p$. The bottom right entry is rewritten as $s_d s_p \det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)$. |

- Set $\mathtt{R}' \leftarrow \mathtt{V}'_d \mathtt{Z}' \mathtt{V}'^\top_p$

By replacing $\mathtt{Z}'$ by its expression directly above, and using equations (39) and (52), we obtain:

$$\mathtt{R}' = \mathtt{V}_d \,\text{diag}(\mathtt{C}_d, s_d)\,\text{diag}\left(\mathtt{C}_d^\top \mathtt{U}\mathtt{C}_p, s_d s_p \det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right)\text{diag}(\mathtt{C}_p^\top, s_p)\mathtt{V}_p^\top = \mathtt{V}_d \mathtt{Z}\mathtt{V}_p^\top = \mathtt{R}.$$

In case 2–2 the solution is thus unique. The optimal cost vanishes, which means that this is an instance of the exact weak-perspective case.

**Case 2–3:** $\sigma_{p,1} = \sigma_{p,2} > 0$**,** $\sigma_{d,1} > \sigma_{d,2} = 0$**.** We define $\sigma_p \stackrel{\text{def}}{=} \sigma_{p,1} = \sigma_{p,2}$.

- Set $(\mathtt{U}'_p, \Sigma'_p, \mathtt{V}'_p) \leftarrow \text{SVD}(\mathtt{P})$, $\Sigma'_p = [\text{diag}(\sigma'_{p,1}, \sigma'_{p,2}) \;\; \mathbf{0}]$

| |
|---|
| See case 2–1. |

- Set $(\mathtt{U}'_d, \Sigma'_d, \mathtt{V}'_d) \leftarrow \text{SVD}(\mathtt{D})$, $\Sigma'_d = [\text{diag}(\sigma'_{d,1}, \sigma'_{d,2}) \;\; \mathbf{0}]$

| |
|---|
| See case 1–3 |

- Set $\mathtt{U}' \leftarrow \mathtt{U}'^\top_d \mathtt{U}'_p$

From equations (44) and (50) we obtain:

$$\mathtt{U}' = \mathtt{E}_d \,\text{diag}(s_{d,1}, s_{d,2})\mathtt{U}\mathtt{C}_p. \quad (56)$$

- Set $\beta' \leftarrow \sigma'_{d,1}\sigma'_{p,1} + \sigma'_{d,2}\sigma'_{p,2}$

| |
|---|
| From equations (47), (48) and (53) we obtain $\beta' = \sigma_p \sigma_{d,1} = \beta$. |

- Set $\gamma' \leftarrow \sigma'_{d,1}\sigma'_{p,2} + \sigma'_{d,2}\sigma'_{p,1}$

| |
|---|
| From equations (47), (48) and (53) we obtain $\gamma' = \sigma_p \sigma_{d,1} = \gamma$. We note that $\gamma' = \gamma = \beta = \beta'$. |

- Set $\delta' \leftarrow \sigma'^2_{d,1} + \sigma'^2_{d,2}$

| |
|---|
| From equations (47) and (48) we obtain $\delta' = \sigma'^2_{d,1} = \sigma^2_{d,1} = \delta$. |

- Set $\eta' \leftarrow \sqrt{\beta'^2 u'^2_{2,2} + \gamma'^2 u'^2_{1,2}}$

| |
|---|
| See case 1–2 and case 2–1, $\eta' = \eta$ and $\eta = \gamma = \beta$. |

- Set $\alpha' \leftarrow \frac{\eta'}{\delta'}$

| |
|---|
| We have $\alpha' = \frac{\eta}{\delta} = \alpha$ since $\eta' = \eta$ and $\delta' = \delta$. |

- Set $\mathtt{Z}' \leftarrow \text{diag}\left(\frac{1}{\eta'}[\beta', \gamma']_\times \odot \mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right)$

| |
|---|
| Following case 1–2, and using equations (46), (52) and (56), the leading block simplifies to $\mathtt{U}' = \mathtt{E}_d \,\text{diag}(s_{d,1}, s_{d,2})\mathtt{U}\mathtt{C}_p$. The bottom right entry is rewritten $s_{d,2} s_p \det(\mathtt{C}_d)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)$. |

- Set $\mathtt{R}' \leftarrow \mathtt{V}_d'\mathtt{Z}'\mathtt{V}_p'^\top$

By replacing $\mathtt{Z}'$ by its expression directly above, and using equations (46) and (52), we obtain:

$$\begin{aligned}\mathtt{R}' &= \mathtt{V}_d \operatorname{diag}(s_{d,1}, \mathtt{C}_d) \operatorname{diag}(\mathtt{E}_d, 1)\\ &\quad \operatorname{diag}\left(\mathtt{E}_d \operatorname{diag}(s_{d,1}, s_{d,2})\mathtt{U}\mathtt{C}_p, s_{d,2}s_p \det(\mathtt{C}_d)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right) \operatorname{diag}(\mathtt{C}_p^\top, s_p)\mathtt{V}_p^\top\\ &= \mathtt{V}_d \operatorname{diag}(1, \mathtt{C}_d) \operatorname{diag}\left(\operatorname{diag}(1, s_{d,2})\mathtt{U}, s_{d,2}\det(\mathtt{C}_d)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right)\mathtt{V}_p^\top\\ &= \mathtt{V}_d \operatorname{diag}(1, \mathtt{G}_d)\mathtt{Z}\mathtt{V}_p^\top,\end{aligned}$$

with, as in case 1–3, $\mathtt{G}_d \in \mathbb{SO}_2$, $\mathtt{G}_d \stackrel{\text{def}}{=} s_{d,2}\mathtt{C}_d \operatorname{diag}(1, \det(\mathtt{C}_d))$.

In case 1–3, the estimated scale $\alpha$ is thus unique, but the rotation $\mathtt{R}$ has an ambiguity of type $\mathcal{A}_{\text{D}} \equiv \mathbb{SO}_2$. Choosing $\mathtt{G}_d = \mathtt{I}$ leads to $\mathtt{R}' = \mathtt{R}$, implying that the solution $\mathtt{R}$ returned by the algebraic procedure always lies in the solution space. The optimal cost simplifies to $\sigma_p^2$.

***Case 2–4: $\sigma_{p,1} = \sigma_{p,2} > 0$, $\sigma_{d,1} = \sigma_{d,2} = 0$.** In case 2–4, $\mathtt{D} = \mathtt{0} \in \mathbb{R}^{2\times 3}$, and no information can be retrieved on the scale and the rotation, since they both vanish from the cost function, as studied in §5.4. The overall ambiguity space is thus $\mathbb{SO}_3 \times \mathbb{R}^+$. Defining $\sigma_p \stackrel{\text{def}}{=} \sigma_{p,1} = \sigma_{p,2}$, the optimal cost is $2\sigma_p^2$.

**Case 3–1: $\sigma_{p,1} > \sigma_{p,2} = 0$, $\sigma_{d,1} > \sigma_{d,2} > 0$.** In case 3–1 almost all the early steps are similar to case 1–1. In fact, they could be further simplified owing to $\sigma_{p,2} = 0$, but this is not necessary for the demonstration. We thus here only give the steps which differ from case 1–1.

- Set $(\mathtt{U}_p', \Sigma_p', \mathtt{V}_p') \leftarrow \text{SVD}(\mathtt{P})$, $\Sigma_p' = [\operatorname{diag}(\sigma_{p,1}', \sigma_{p,2}')\ \ \mathbf{0}]$

From table 2 we have, for $\mathtt{C}_p \in \mathbb{O}_2$ and $s_{p,1}, s_{p,2} \in \{-1, 1\}$:

$$\mathtt{U}_p' = \mathtt{U}_p \operatorname{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p \quad (57)$$
$$\Sigma_p' = \mathtt{E}_p\Sigma_p \operatorname{diag}(\mathtt{E}_p, 1) \quad (58)$$
$$\mathtt{V}_p' = \mathtt{V}_p \operatorname{diag}(s_{p,1}, \mathtt{C}_p)\operatorname{diag}(\mathtt{E}_p, 1) \quad (59)$$
$$\sigma_{p,1}' = e_p\sigma_{p,1} \quad (60)$$
$$\sigma_{p,2}' = (1 - e_p)\sigma_{p,1}. \quad (61)$$

- Set $\mathtt{Z}' \leftarrow \operatorname{diag}\left(\frac{1}{\eta'}[\beta', \gamma']_\times \odot \mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}_d')\det(\mathtt{V}_p')\right)$

The leading block is expanded as in case 1–1. From equations (25), (29) and (59) we simplify the bottom right entry to:

$$s_{d,3}s_{p,2}\det(\mathtt{C}_p)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p). \quad (62)$$

- Set $\mathtt{R}' \leftarrow \mathtt{V}_d'\mathtt{Z}'\mathtt{V}_p'^\top$

Using equations (25), (35), (59) and (62) we obtain:

$$\begin{aligned}\mathtt{R}' &= \mathtt{V}_d \operatorname{diag}(s_{d,1}, s_{d,2}, s_{d,3}) \operatorname{diag}(\mathtt{E}_d, 1)\\ &\quad \operatorname{diag}\left(\frac{1}{\eta}\mathtt{E}_d([\beta, \gamma]_\times \odot (\operatorname{diag}(s_{d,1}, s_{d,2})\mathtt{U}\operatorname{diag}(s_{p,1}, s_{p,2})))\mathtt{E}_p, s_{d,3}s_{p,2}\det(\mathtt{C}_p)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right)\\ &\quad \operatorname{diag}(\mathtt{E}_p)\operatorname{diag}(s_{p,1}, \mathtt{C}_p^\top)\mathtt{V}_p^\top\\ &= \mathtt{V}_d \operatorname{diag}\left(\frac{1}{\eta}[\beta, \gamma]_\times \odot \mathtt{U}\operatorname{diag}(1, s_{p,2}), s_{p,2}\det(\mathtt{C}_p)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right)\operatorname{diag}(1, \mathtt{C}_p^\top)\mathtt{V}_p^\top\\ &= \mathtt{V}_d\mathtt{Z}\operatorname{diag}(1, \mathtt{G}_p^\top)\mathtt{V}_p^\top,\end{aligned}$$

with $\mathtt{G}_p \in \mathbb{SO}_2$, $\mathtt{G}_p \stackrel{\text{def}}{=} s_{p,2}\mathtt{C}_p \operatorname{diag}(1, \det(\mathtt{C}_p))$.

In case 3–1, the estimated scale $\alpha$ is thus unique, but the rotation $\mathtt{R}$ has an ambiguity of type $\mathcal{A}_{\text{P}} \equiv \mathbb{SO}_2$. Choosing $\mathtt{G}_p = \mathtt{I}$ leads to $\mathtt{R}' = \mathtt{R}$, implying that the solution $\mathtt{R}$ returned by the algebraic procedure always lies in the solution space. The optimal cost simplifies to $\frac{\sigma_{p,1}^2}{\sigma_{d,1}^2+\sigma_{d,2}^2}(\sigma_{d,1}^2u_{1,2}^2 + \sigma_{d,2}^2u_{2,2}^2)$.

**Case 3–2: $\sigma_{p,1} > \sigma_{p,2} = 0$, $\sigma_{d,1} = \sigma_{d,2} > 0$.** In case 3–2 almost all the early steps are similar to case 1–2. In fact, they could be further simplified owing to $\sigma_d \stackrel{\text{def}}{=} \sigma_{d,1} = \sigma_{d,2}$, but this is not necessary for the demonstration. We thus only give the steps which differ from case 1–2.

- Set $(\mathtt{U}'_p, \Sigma'_p, \mathtt{V}'_p) \leftarrow \text{SVD}(\mathtt{P})$, $\Sigma'_p = [\text{diag}(\sigma'_{p,1}, \sigma'_{p,2})\ \ \mathbf{0}]$

  See case 3–1.

- Set $(\mathtt{U}'_d, \Sigma'_d, \mathtt{V}'_d) \leftarrow \text{SVD}(\mathtt{D})$, $\Sigma'_d = [\text{diag}(\sigma'_{d,1}, \sigma'_{d,2})\ \ \mathbf{0}]$

  See case 1–2

- Set $\mathtt{Z}' \leftarrow \text{diag}\left(\frac{1}{\eta'}[\beta', \gamma']_\times \odot \mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right)$

  The leading block is expanded as in case 1–2. From equations (39), (41) and (59), we simplify the bottom right entry to:

  $$s_d s_{p,2} \det(\mathtt{C}_p)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p). \tag{63}$$

- Set $\mathtt{R}' \leftarrow \mathtt{V}'_d \mathtt{Z}' \mathtt{V}'^\top_p$

  Using equations (39), (42), (59) and (63) we obtain:

  $$\begin{aligned} \mathtt{R}' &= \mathtt{V}_d\,\text{diag}(\mathtt{C}_d, s_d)\,\text{diag}\left(\mathtt{C}_d^\top \mathtt{U}\,\text{diag}(s_{p,1}, s_{p,2})\mathtt{E}_p, s_d s_{p,2}\det(\mathtt{C}_p)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right) \\ &\quad \text{diag}(\mathtt{E}_p, 1)\,\text{diag}(s_{p,1}, \mathtt{C}_p^\top)\mathtt{V}_p^\top \\ &= \mathtt{V}_d \mathtt{Z}\,\text{diag}(1, \mathtt{G}_p^\top)\mathtt{V}_p^\top, \end{aligned}$$

  with $\mathtt{G}_p \in \mathbb{SO}_2$, $\mathtt{G}_p \stackrel{\text{def}}{=} s_{p,2}\mathtt{C}_p\,\text{diag}(1, \det(\mathtt{C}_p))$.

In case 3–2, the estimated scale $\alpha$ is thus unique, but the rotation $\mathtt{R}$ has an ambiguity of type $\mathcal{A}_{\text{P}} \equiv \mathbb{SO}_2$. Choosing $\mathtt{G}_p = \mathtt{I}$ leads to $\mathtt{R}' = \mathtt{R}$, implying that the solution $\mathtt{R}$ returned by the algebraic procedure always lies in the solution space. The optimal cost simplifies to $\frac{\sigma_{p,1}^2}{2}$.

**Case 3–3: $\sigma_{p,1} > \sigma_{p,2} = 0$, $\sigma_{d,1} > \sigma_{d,2} = 0$.** In case 3–3 almost all the early steps are similar to case 1–1. In fact, they could be further simplified owing to $\sigma_{p,2} = \sigma_{d,2} = 0$, but this is not necessary for the demonstration. We thus only give the steps which differ from case 1–1.

- Set $(\mathtt{U}'_p, \Sigma'_p, \mathtt{V}'_p) \leftarrow \text{SVD}(\mathtt{P})$, $\Sigma'_p = [\text{diag}(\sigma'_{p,1}, \sigma'_{p,2})\ \ \mathbf{0}]$

  See case 3–1.

- Set $(\mathtt{U}'_d, \Sigma'_d, \mathtt{V}'_d) \leftarrow \text{SVD}(\mathtt{D})$, $\Sigma'_d = [\text{diag}(\sigma'_{d,1}, \sigma'_{d,2})\ \ \mathbf{0}]$

  See case 1–3

- Set $\mathtt{Z}' \leftarrow \text{diag}\left(\frac{1}{\eta'}[\beta', \gamma']_\times \odot \mathtt{U}', \det(\mathtt{U}')\det(\mathtt{V}'_d)\det(\mathtt{V}'_p)\right)$

  The leading block is expanded as in case 1–1. From equations (29), (46) and (59) we simplify the bottom right entry to:

  $$s_{d,2} s_{p,2} \det(\mathtt{C}_d)\det(\mathtt{C}_p)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p). \tag{64}$$

- Set $\mathtt{R}' \leftarrow \mathtt{V}'_d \mathtt{Z}' \mathtt{V}'^\top_p$

  Using equations (35), (46), (59) and (64) we obtain:

  $$\begin{aligned} \mathtt{R}' &= \mathtt{V}_d\,\text{diag}(s_{d,1}, \mathtt{C}_d)\,\text{diag}(\mathtt{E}_d, 1) \\ &\quad \text{diag}\left(\frac{1}{\eta}\mathtt{E}_d([\beta, \gamma]_\times \odot (\text{diag}(s_{d,1}, s_{d,2})\mathtt{U}\,\text{diag}(s_{p,1}, s_{p,2})))\mathtt{E}_p, s_{d,2}s_{p,2}\det(\mathtt{C}_d)\det(\mathtt{C}_p)\det(\mathtt{U})\det(\mathtt{V}_d)\det(\mathtt{V}_p)\right) \\ &\quad \text{diag}(\mathtt{E}_p, 1)\,\text{diag}(s_{p,1}, \mathtt{C}_p^\top)\mathtt{V}_p^\top \\ &= \mathtt{V}_d\,\text{diag}(1, \mathtt{G}_d)\mathtt{Z}\,\text{diag}(1, \mathtt{G}_p^\top)\mathtt{V}_p^\top, \end{aligned}$$

  with $\mathtt{G}_d, \mathtt{G}_p \in \mathbb{SO}_2$, $\mathtt{G}_d \stackrel{\text{def}}{=} s_{d,2}\mathtt{C}_d\,\text{diag}(1, \det(\mathtt{C}_d))$ and $\mathtt{G}_p \stackrel{\text{def}}{=} s_{p,2}\mathtt{C}_p\,\text{diag}(1, \det(\mathtt{C}_p))$.

In case 3–3, the estimated scale $\alpha$ is thus unique, but the rotation $\mathtt{R}$ has an ambiguity of type $\mathcal{A}_{\text{DP}} \equiv \mathbb{SO}_2^2$. Choosing $\mathtt{G}_d = \mathtt{G}_p = \mathtt{I}$ leads to $\mathtt{R}' = \mathtt{R}$, implying that the solution $\mathtt{R}$ returned by the algebraic procedure always lies in the solution space. The optimal cost simplifies to $\sigma_{p,1}^2 u_{1,2}^2$.

***Case 3–4: $\sigma_{p,1} > \sigma_{p,2} = 0$, $\sigma_{d,1} = \sigma_{d,2} = 0$.** In case 3–4, $\mathtt{D} = \mathtt{0} \in \mathbb{R}^{2\times 3}$, and no information can be retrieved on the scale and the rotation, since they both vanish from the cost function, as studied in §5.4. The overall ambiguity space is thus $\mathbb{SO}_3 \times \mathbb{R}^+$. The optimal cost is $\sigma_{p,1}^2$.

***Cases 4–{1,2,3}: $\sigma_{p,1} = \sigma_{p,2} = 0$, $\sigma_{d,1} > 0$, $\sigma_{d,1} \geq \sigma_{d,2} \geq 0$.** In cases 4–1, 4–2 and 4–3 $\mathtt{P} = \mathtt{0} \in \mathbb{R}^{2\times 3}$. No information can be recovered on the rotation, but the optimal scale is $\alpha' = \alpha = 0$, as studied in §5.4. The overall ambiguity space is thus $\mathbb{SO}_3$. The optimal cost vanishes.

***Case 4–4: $\sigma_{p,1} = \sigma_{p,2} = 0$, $\sigma_{d,1} = \sigma_{d,2} = 0$.** In case 4–4, $\mathtt{P} = \mathtt{D} = \mathtt{0} \in \mathbb{R}^{2\times 3}$, and no information can be retrieved on the scale and the rotation, since they both vanish from the cost function, as studied in §5.4. The overall ambiguity space is thus $\mathbb{SO}_3 \times \mathbb{R}^+$. The optimal cost vanishes.